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THE PRINCIPLES OF MATHEMATICS AND NATURAL PHILOSOPHY.

IN FOUR VOLUMES.

VOL. III. PART I.

BRITISH MUSEUM

THE
PRINCIPLES OF MECHANICS:
DESIGNED
FOR THE USE OF STUDENTS
IN THE
UNIVERSITY.

W

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THE PRINCIPLES OF MECHANICS.

THE term *Mechanics* has at different times, and by different writers, been applied to branches of ſcience eſſentially diſtinct from each other. It was originally confined to the doctrine of equilibrium, or the inveſtigation of the proportion of powers when they balance each other.

Later writers, adapting the term to their diſcoveries, have uſed it to denote that ſcience with treats of the nature, production, and alteration of motion; giving to the former branch, by way of contradiſtinction, the name of ſtatics.

Others, giving the term a ſtill more comprehenſive meaning, have applied it to both theſe ſciences.

None of theſe definitions will exactly ſuit our preſent purpoſe; the firſt being too contracted, and the others much too extenſive, for a treatiſe which is intended to be an introduction to the higher branches of philoſophy. Our ſyſtem of mechanics will compriſe the doctrine of equilibrium, and ſo much of the ſcience of motion as is neceſſary to explain the effects of impact and gravity.

SECTION

SECTION I.

ON MATTER AND MOTION.

DEFINITIONS.

ART. 1. *MATTER* is a ſubſtance, the object of our ſenſes, in which are always united the following properties; *extenſion*, *figure*, *ſolidity*, *mobility*, *diviſibility*, *gravity*, and *inactivity*.

2. *Extenſion* may be conſidered in three points of view: 1ſt. As ſimply denoting the part of ſpace which lies between two points, in which caſe it is called *diſtance*. 2d. As implying both length and breadth, when it is denominated *ſurface* or *area*. 3d. As compriſing three dimenſions, length, breadth, and thickneſs, in which caſe it may be called *bulk*, *capacity* or *content*. It is uſed in the laſt of theſe ſenſes when it is ſaid to be a property of matter.

3. *Figure* is the boundary of extenſion. The portions of matter, from which we receive our ideas of this ſubſtance, are bounded; that is, they have figure.

4. *Solidity* is that property of matter by which it fills ſpace; or, by which any portion of matter ex-

cludes every other portion from that part of ſpace which it occupies; and thus it is capable of reſiſtance and protruſion. "There is no idea which we receive more conſtantly from ſenſation than *ſolidity*. Whether we move or reſt, in what poſture ſoever we are, we always feel ſomething under us that ſupports us, and hinders our farther ſinking downwards; and the bodies which we daily handle make us perceive that, whilſt they remain between them, they do by an inſurmountable force hinder the approach of the parts of our hands that preſs them *."

5. *Mobility*, or a capacity of being transferred from one place to another, in a quality found to belong to all bodies upon which we can make ſuitable experiments; and hence we conclude that it belongs to all matter.

6. *Diviſibility* ſignifies a capacity of being ſeparated into parts. That matter is thus diviſible, our daily experience aſſures us. How far the diviſion can actually be carried is not ſo eaſily ſeen. We know that many bodies may be reduced to a very fine powder by trituration; by chemical ſolution, the parts of a body may be ſo far divided as not to be ſenſible to the ſight; and by the help of the microſcope we diſcover myriads of organized bodies, totally unknown before ſuch inſtruments were invented. Theſe and ſimilar conſiderations, lead us to conclude, that the diviſion of matter is carried to a degree of minuteneſs far exceeding the bounds of our faculties; and it ſeems not unreaſonable to ſuppoſe, that this capacity of diviſion is without limit; eſpecially, as we can

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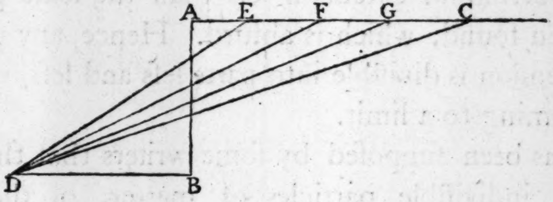
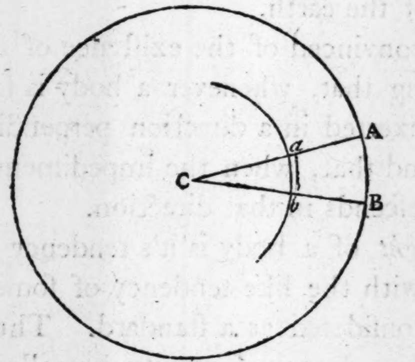
* LOCKE's Eſſay. B. II. Ch. IV.

prove, theoretically, that any portion of extenſion is diviſible into parts leſs and leſs without end [*].

From the extremities of the line *AB*, draw *AC*, *BD*, parallel to each other, and in oppoſite directions; in *AC* take any number of points *E*, *F*, *G*, &c. and join *DE*, *DF*, *DG*, &c. theſe lines will cut *AB* in different points; and ſince, in the indefinite line *AC*, an unlimited number of points may be taken, the number of parts into which *AB* is diviſible, is indefinite.



This property of extenſion may alſo be proved ex abſurdo. If poſſible, let *AB* be the leaſt portion of a circular



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[*] Porro corporum partes diviſas et ſibi mutuò contiguas ab invicem ſeparari poſſe ex phænominis novimus, et partes indiviſas in partes minores ratione diſtingui poſſe ex mathematicâ certum eſt.

circular arc; take *C* the center, join *AC*, *CB*, and with the center *C*, and any radius *Ca*, leſs than *CA*, deſcribe a circle cutting *CA* and *CB* in the points *a* and *b*; then becauſe *AB* and *ab* are ſimilar arcs, they are as their radii; therefore *ab* is leſs than *AB*; or a portion of extenſion leſs than the leaſt poſſible has been found, which is abſurd. Hence, any portion of extenſion is diviſible into parts leſs and leſs, without ever coming to a limit.

It has been ſuppoſed by ſome writers that there are certain indiviſible particles of matter, of the ſame figure and dimenſions, by the different modifications of which different bodies are formed. As no arguments are adduced in favour of this hypotheſis, and as experiment ſeems to lead us to a contrary concluſion, we cannot allow it a place amongſt the principles upon which our theory is to reſt.

7. *Gravity* is the tendency which all bodies have to the center of the earth.

We are convinced of the exiſtence of this tendency by obſerving that, whenever a body is ſuſtained, it's preſſure is exerted in a direction perpendicular to the horizon; and that, when the impediment is removed, it always deſcends in that direction.

The *weight* of a body is it's tendency to the earth, compared with the like tendency of ſome other body, which is conſidered as a ſtandard. Thus, if a body with a certain degree of gravity be called one pound, any other body which has the ſame degree of gravity, or

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eſt. Utrum verò partes illæ diſtinctæ et nondum diviſæ per vires naturæ dividi et ab invicem ſeparari poſſint, incertum eſt. Newt. Princip. L. III. Reg. 3.

or which by it's gravity will produce the ſame effect, under the ſame circumſtances, is alſo called a pound; and theſe two together, two pounds, &c.

Gravity is not an accidental property of matter ariſing from the figure or diſpoſition of the parts of a body; for then, by changing it's ſhape, or altering the arrangement of the particles which compoſe it, the gravitation of the maſs would be altered. But we find that no ſeparation of the particles, no change of the ſtructure, which human power can effect, produces any alteration in the weight.

As gravity is a property belonging to every particle of a body, independent of it's ſituation with reſpect to other paticles, the gravity of the whole is the aggregate of the gravities of all it's parts. Thus, though the weight of the whole is not altered by any diviſion, or new arrangement of the particles, yet every increaſe or diminution of their number, produces a correſponding increaſe or diminution of the weight.

Our preſent ſubject does not lead us to conſider gravitation in any other point of view than ſimply as a tendency in bodies to the center of the earth, or to attend to it's effects at any conſiderable diſtance from the ſurface; it may not, however, be improper to obſerve that the operation of this principle is much more extenſive. Every portion of matter gravitates towards every other portion, in that part of the ſyſtem of nature which falls under our obſervation. The gravitation indeed of ſmall particles towards each other is not perceived, on account of the ſuperior action of the earth*; yet it has been found, by the accurate obſervations of Dr. Maskelyne, in Scotland, that the attraction of a mountain is ſufficient to draw the plumb-line ſenſibly from the perpendicular.

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* This attraction is not ſufficient to explain the common experiment

Sir I. Newton has diſcovered that the moon is retained in her orbit by the agency of a cauſe ſimilar to that by which a body falls to the ground, differing from it only in degree, and this in conſequence of the greater diſtance of the moon from the earth's center. The ſame author has demonſtrated that the planets are retained in their reſpective orbits by a principle of the ſame kind; and that the minuteſt irregularities in their motions may be ſatisfactorily deduced from the known laws of it's operation.

8. *Inactivity* may be conſidered in two lights: 1ſt. As an *inability* in matter to change it's ſtate of reſt or uniform rectilinear motion: 2d. As that quality by which it *reſiſts* any ſuch change†. In this latter ſenſe it is uſually called the *force of inactivity*, the *inertia*, or the *vis inertiæ*.

The inactivity of matter, according to the former explanation, is laid down as a law of motion; the truth of which we ſhall endeavour to eſtabliſh in the next ſection.

That

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ment of two particles of the ſame kind, as quickſilver, &c. when placed upon a ſmooth horizontal plane, running together. If the effect were not produced by ſome power different from gravitation, a drop of oil would run in the ſame manner towards a drop of water, which is not found to be the caſe.

† That is, a change from reſt to uniform rectilinear motion, or a change in it's uniform rectilinear motion. The term is ſometimes applied to the reſiſtance which a body makes to the production, or alteration of rotatory motion; this, in reality, is the effect of the inertia acting at a mechanical advantage, or diſadvantage.

That a body *reſiſts* any change in it's ſtate of reſt, or uniform rectilinear motion, is known from conſtant experience. We cannot move the leaſt particle of matter without ſome exertion; nor can we deſtroy any motion without perceiving ſome reſiſtance*. Thus we ſee, in general, that *inertia* is a property inherent in all bodies with which we are concerned; different indeed in different caſes, but exiſting, in a greater or leſs degree, in all. The quantity we are not at preſent conſidering; the exiſtence of the property, every one, from his own obſervation, will readily allow. We know indeed from experience †, that the inertia of a body is not altered by altering the arrangement of it's parts; but if one portion of matter be added to another, the inertia of the whole is increaſed; and if any part be removed, the inertia is leſſened. This clearly ſhews that it exiſts, independently, in every particle, and that the whole inertia is the aggregate of all it's parts.

Hence it follows, from our notion of quantity, that if to a body with a certain quantity of inertia, another, which has an equal quantity, be added, the whole inertia will be doubled; and that by the repeated addition of equal quantities, the whole inertia will be increaſed in the ſame proportion with the number of parts.

Theſe properties, which are always found to exiſt together in the ſame ſubſtance, have ſometimes been ſaid to be eſſential to matter: Whether they are, or are not *neceſſarily* united in the ſame ſubſtance it is impoſſible to decide, nor does it concern us to enquire. The buſineſs of natural philoſophy is not to find out what might have been the conſtitution of nature, but to examine what it is in fact; and to account for the phænomena, which fall under our obſervation, from thoſe properties of matter which we know by experience that it poſſeſſes.

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* It muſt be obſerved, that this reſiſtance is diſtinct from, and independent of gravity; becauſe it is perceived where gravity produces no effect; as, when a wheel is turned round it's axis, or a body moved along an horizontal plane.

† Vid. Art. 25.

9. By the *quantity of matter* in a body, we underſtand the aggregate of it's particles, each of which has a certain degree of inertia. Or, in other words, if we ſuppoſe bodies made up of particles, each of which has the ſame inertia, the quantity of matter in one, is to the quantity of matter in another, as the number of ſuch particles in the former body, to the number in the latter*.

When we conſider bodies as made up of parts, and compare them in this reſpect, it becomes neceſſary to give a definite and preciſe deſcription of thoſe parts; otherwiſe our notion of the quantity will be vague and inaccurate. Now the only properties of matter which admit of exact compariſon, and which depend upon the number, and not upon the arrangement of the particles, are weight and inertia; either of which may properly be made uſe of as a meaſure of the *quantity* of matter; and ſince, at a given place, they are proportional to each other, as we ſhall ſhew hereafter (Art. 25), it is of little conſequence which meaſure we adopt. The inertia has been fixed upon, becauſe the gravity of a body, though invariable at the ſame place, is different at different diſtances from the center of the earth; whereas, the inertia is always, and under all circumſtances, the ſame.

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* Quantitas materiæ eſt menſura ejuſdem orta ex illius denſitate et magnitudine conjunctim. NEWT. Princip. Def. 1.

Ejuſdem eſſe denſitatis dico, quarum vires inertiæ ſunt ut magnitudines. Lib. III. Prop. 6. Cor. 4.

Attendi enim oportet ad punctorum numerum, ex quibus corpus movedum eſt conflatum. Puncta vero ea inter ſe æqualia cenſeri debent, non quæ æque ſunt parva, ſed in quæ eadem potentia æquales exerit effectus EUL. Mech. 139.

The *denſity* of a body is meaſured by the quantity of matter in a *given bulk*.

10. By *motion* we underſtand a change of place; and it is of two kinds, *abſolute* and *relative*.

A body is ſaid to be in *abſolute motion* when it is actually transferred from one point in fixed ſpace to another; and to be *relatively in motion*, when it's ſituation is changed with reſpect to the ſurrounding bodies.

Theſe two kinds of motion evidently coincide when the bodies, to which the reference is made, happen to be fixed. In other caſes, a body relatively in motion, or relatively at reſt, may or may not be abſolutely in motion. Thus, a ſpectator ſtanding ſtill on the ſhore, if his place be referred to a ſhip which ſails by, is relatively in motion; and the ſeveral parts of the veſſel are at reſt, with reſpect to each other, though the whole is transferred from one part of ſpace to another.

When a body always paſſes over equal parts of ſpace in equal ſucceſſive portions of time, it's motion is ſaid to be *uniform*. When the ſucceſſive portions of ſpace, deſcribed in equal times, continually increaſe, the motion is ſaid to be *accelerated*; and to be *retarded*, when thoſe ſpaces continually decreaſe. Alſo the motion is ſaid to be *uniformly* accelerated or retarded, when the increments or decrements of the ſpaces, deſcribed in equal ſucceſſive portions of time, are always equal.

11. The *velocity* of a body, or *rate* of it's motion, is meaſured by the ſpace, *uniformly deſcribed* in a *given* time.

The given time, taken as a ſtandard, is uſually one ſecond; and the ſpace deſcribed is meaſured in feet. Thus, when v repreſents a body's velocity, v is the number of feet which the body would uniformly deſcribe in one ſecond.

If a body's motion be accelerated or retarded, the velocity at any point is not meaſured by the ſpace actually deſcribed in a given time, but by the ſpace which would have been deſcribed in the given time, if the motion had continued uniform, from that point

12. Cor. 1. If two bodies move uniformly on the ſame line, in *oppoſite* directions, their relative velocity is equal to the *ſum* of their abſolute velocities, ſince the ſpace by which they uniformly approach to, or recede from, each other, in any time, is equal to the ſum of the ſpaces which they reſpectively deſcribe in that time.

When the bodies move in the *ſame* direction, their relative velocity is equal to the *difference* of their abſolute velocities.

13. Cor. 2. When a body moves with an uniform velocity, the ſpace deſcribed is proportional to the time of it's motion.

Let the body deſcribe a feet in one ſecond, then ſince the motion is uniform, it will deſcribe ta feet in t ſeconds; that is, the ſpace deſcribed is proportional to the time.

14. Cor. 3. When bodies have different uniform motions, the ſpaces deſcribed are proportional to the times and velocities jointly *.

Let V and v be the velocities of two bodies A and B; T and t the times of their motions; S and s the ſpaces deſcribed. Alſo let S' be the ſpace deſcribed by B in the time T:

$$\begin{array}{rl} \text{Then } & S : S' :: V : v \text{ (Art. 11)}, \\ & \underline{S' : s :: T : t \text{ (Art. 13)},} \\ \text{Comp.} & S : s :: TV : tv; \end{array}$$

that is, $S \propto TV$ (Alg. Art. 195).

Ex. Let the times be to each other as $6 : 5$, and the velocities as $2 : 3$; then $S : s :: 2 \times 6 : 3 \times 5 :: 4 : 5$.

15. Cor. 4. Since $S \propto TV$, we have $V \propto \frac{S}{T}$, and $T \propto \frac{S}{V}$, (Alg. Art. 205).

Ex. 1. Let A move uniformly through 5 feet in $3''$, and B through 9 feet in $7''$; required the ratio of the velocities.

$$V : v :: \frac{5}{3} : \frac{9}{7} :: 35 : 27.$$

Ex. 2.

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* Since the times and velocities may in each caſe be repreſented by numbers, there is no impropriety in ſpeaking of their products. The truth of this obſervation will be evident, if the propoſition be expreſſed in different words: When the uniform velocities of two bodies are in the ratio of the numbers V and v, and the times of their motions in the ratio of the numbers T and t, the ſpaces deſcribed are in the ratio of the numbers TV and tv.

Ex. 2. Let *A*'s velocity be to *B*'s velocity as 5 to 4; to compare the times in which they will deſcribe 9 and 7 feet reſpectively.

$$T : t :: \frac{9}{5} : \frac{7}{4} :: 36 : 35.$$

16. Cor. 5. Since the areas of rectangles are in the ratio compounded of the ratios of their ſides, if the baſes repreſent the velocities of two motions, and altitudes the times, the areas will repreſent the ſpaces deſcribed.

17. The *quantity of motion*, or *momentum*, of a body, is meaſured by the velocity and quantity of matter jointly.

Thus, if the quantities of matter in two bodies be repreſented by 6 and 7, and their velocities by 9 and 8, the ratio of 6×9 to 7×8, or 27 to 28, is *called* the ratio of their *momenta*.

18. Cor. 1. If M be the momentum of a body, Q it's quantity of matter, and V it's velocity, then ſince $M \propto QV$, we have $Q \propto \frac{M}{V}$; and $V \propto \frac{M}{Q}$.

Ex. If the quantities of motion be as 6 to 5, and the velocities as 7 to 8, what is the ratio of the quantities of matter?

Since $Q \propto \frac{M}{V}$, we have $Q : q :: \frac{6}{7} : \frac{5}{8} :: 48 : 35$.

19. Cor. 2. If M be given, $Q \propto \frac{1}{V}$; and conſequently if $Q \propto \frac{1}{V}$, M is invariable. (Algebra, Art. 206).

20. Whatever changes, or tends to change, the ſtate of reſt or uniform rectilinear motion of a body, is called *force*.

Thus, preſſure, impact, gravity, &c. are called forces.

When a force produces it's effect inſtantaneouſly, it is ſaid to be *impulſive**. When it acts inceſſantly, it is called a *conſtant* force.

Conſtant forces are of two kinds, *uniform* and *variable*. A force is ſaid to be *uniform* when it always produces *equal* effects in equal ſucceſſive portions of time; and *variable*, when the effects produced in equal times are *unequal*.

Forces which are known to us only by their effects, muſt be compared by eſtimating thoſe effects under the ſame circumſtances. Thus, impulſive forces muſt be meaſured by the whole effects produced; uniform forces, by the effects produced in equal times; and variable forces, by the effects which would be produced in equal times, were they to become and continue uniform during thoſe times.

The effects produced by the actions of forces are of two kinds, velocity and momentum; and thus we have two methods of comparing them, according as we conceive them to be the cauſes of velocity or momentum.

21. The *accelerating force* is meaſured by the *velocity* uniformly generated in a given time, no regard being had to the quantity of matter moved.

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* Though we cannot conceive finite effects to be produced otherwiſe than by degrees, and conſequently in ſucceſſive portions of time; yet when theſe portions are ſo ſmall as not to be diſtinguiſhable by our faculties, the effects may be ſaid to be inſtantaneous.

Thus, if the velocities uniformly generated, in two caſes, in equal times, be as 6 to 7, the accelerating forces are *ſaid* to be in that ratio.

The accelerating force of gravity at the ſame place is invariable; for all bodies falling freely, in an exhauſted receiver, acquire equal velocities in any given time.

22. The *moving force* is meaſured by the *momentum* uniformly generated in a given time.

If the momenta thus generated, in two caſes, be as 14 to 15, the moving forces are *ſaid* to be in that ratio.

23. Cor. 1. Since the momentum is proportional to the velocity and quantity of matter, the moving force varies as the accelerating force and quantity of matter jointly.

The moving force of gravity varies as the quantity of matter moved, becauſe the accelerating force is given (Art. 21).

24. Cor. 2. Hence it follows that the accelerating force varies as the moving force directly, and the quantity of matter inverſely.

PROP. I.

25. *The vis inertiæ of any body is proportional to it's weight.*

The inertia, as was obſerved on a former occaſion, is the reſiſtance which a body makes to any change in it's ſtate of reſt or uniform rectilinear motion (Art. 8); and this reſiſtance is manifeſtly the ſame in two caſes, if the ſame force, applied in the ſame manner, and for the ſame time, generate the ſame velocity.

Let two bodies *A* and *B*, equal in *weight*, be placed in two ſimilar and equal boxes, which are connected by a ſtring paſſing over a fixed pulley; then theſe will exactly balance each other; and if the whole be put into motion, the gravity can neither accelerate nor retard that motion; the whole reſiſtance therefore to the communication of motion in the ſyſtem, ariſes from the inertia of the weights, the inertia of the ſtring and pulley*, the friction upon the axis, and the reſiſtance of the air†.

Now let a weight *C* be added on one ſide, and let the velocity generated in any given time, in the whole ſyſtem, by this additional weight, be obſerved.

Then in the place of *A*, or *B*, ſubſtitute any other maſs of the ſame weight, and it will be found that *C* will, in the ſame time, generate the ſame velocity in this ſyſtem as in the former; and therefore, the whole reſiſtance to the communication of motion muſt be the ſame. Alſo the inertia of the ſtring and pulley, the friction of the axis, and the air's reſiſtance are the ſame in the two experiments; conſequently, the reſiſtance ariſing from the inertia of the weights is the ſame: That is, ſo long as the weight remains unaltered, whatever be the form or conſtitution of the body, the inertia is the ſame.

Alſo, ſince the whole quantity of inertia is the aggregate inertia of all the parts, if the *weight* be doubled,

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[*] Vid. Note, page 8.

[†] This experiment may be made with great accuracy by means of a machine, invented by Mr. ATWOOD, for the purpoſe of examining the motions of bodies when acted upon by conſtant forces. This machine is deſcribed in his well known treatiſe on the *Rectilinear Motion and Rotation of Bodies*, (p. 299).

doubled, an equal quantity of *inertia* is added to the former quantity, or the whole *inertia* is doubled; and in the ſame manner, if the *weight* be increaſed in any proportion, by the repeated addition of equal *weights*, the *inertia* is increaſed in the ſame proportion.

It may be obſerved, that the velocity generated in a given time, is the ſame, whether the ſyſtem begins to move from reſt or not; therefore the inertia is the ſame, whether the body be at reſt or in motion.

26. Cor. Since the quantity of matter is meaſured by the inertia (Art. 9), it is alſo proportional to the weight.

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SECTION II.

ON THE LAWS OF MOTION.

THE FIRST LAW.

27. *IF a body be at reſt it will continue at reſt, and if in motion it will continue to move uniformly forward in a right line, till it is acted upon by ſome external force.*

That a body at reſt cannot put itſelf in motion we know from conſtant and univerſal experience.

That a body in motion will continue to move uniformly forward in a right line till it is acted upon by ſome external force, though equally certain, is not, it muſt be allowed, equally apparent; ſince all the motions which fall under our immediate obſervation, and rectilinear motions in particular, are ſoon deſtroyed. If however we can point out the cauſes which tend to deſtroy the motions of bodies, and ſhew, experimentally, that by removing ſome of them and diminiſhing others, the motions continually become more uniform and rectilinear, we may juſtly conclude that any

deviation

deviation from the firſt direction, and firſt velocity, muſt be attributed to the agency of external cauſes; and that there is no tendency in matter itſelf, either to increaſe or diminiſh any motion impreſſed upon it.

Now the cauſes which retard a body's motion, beſides colliſion, or the evident obſtruction which it meets with from ſenſible maſſes of matter, are gravity, friction, and the reſiſtance of the air; and it will appear by the following experiments that when theſe are removed, or due allowance is made for their known effects, we are neceſſarily led to infer the truth of the law above laid down.

1ſt. If a ball be thrown along a rough pavement, it's motion, on account of the many obſtacles it meets with, will be very irregular and ſoon ceaſe; but if it be bowled upon a ſmooth bowling-green, it's motion will continue longer, and be more rectilinear; and if it be throwm along a ſmooth ſheet of ice, it will preſerve both it's direction and it's motion for a ſtill longer time.

In theſe caſes, the gravity, which acts in a direction perpendicular to the plane of the horizon, neither accelerates nor retards the motion; the cauſes which produce the latter effect are colliſion, friction, and the air's reſiſtance; and in proportion as the two former of theſe are leſſened, the motion becomes more nearly uniform and rectilinear.

2d. When a wheel is accurately conſtructed, and a rotatory motion about it's axis communicated to it, if the axis and it's ſupporters be well poliſhed, the motion will continue a conſiderable time; if the axis be placed upon friction wheels, the motion will continue longer; and if the apparatus be placed under the receiver of an air pump, and the air be exhauſted, the motion will continue without viſible diminution for a very long time.

In theſe inſtances, gravity, which acts equally on oppoſite points of the wheel, neither accelerates nor retards the motion; and the more care we take to remove the friction, and the reſiſtance of the air, the leſs is the firſt motion diminiſhed in a given time.

3d. If a body be projected in any direction inclined to the horizon, it deſcribes a curve which is nearly the common parabola. This effect is produced by the joint action of gravity and the motion of projection; and ſince the effect produced by the former is known, the effect produced by the latter may be determined. This, it is found, would carry the body uniformly forward in the line in which it was projected; as will fully appear when we come to the doctrine of projectiles. The deviation of the curve deſcribed from the parabolic form is ſufficiently accounted for by the reſiſtance of the air.

From theſe, and ſimilar experiments, we are led to conclude that all bodies in motion would uniformly perſevere in that motion, were they not prevented by external impediments; and that every increaſe or diminution of velocity, every deviation from the line of direction, is to be attributed to the agency of ſuch cauſes.

28. It may not be improper to obſerve, that this law ſuggeſts two methods of diſtinguiſhing between abſolute motions, and ſuch as are only apparent; one, by conſidering the cauſes which produce the motions; and the other, by attending to the effects with which the motions are accompanied*. 1ſt. We

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* Newt. Princip. Schol. ad Def.

1ſt. We may ſometimes diſtinguiſh abſolute motion, or change of abſolute motion, from that which is merely apparent, by conſidering the cauſes which produce them.

When two bodies are abſolutely at reſt, they are relatively ſo; and the appearance is the ſame, when they are moving in the ſame direction, at the ſame rate; a relative motion therefore can only ariſe from an abſolute motion, or change of abſolute motion, in one or both of the bodies. We have ſeen alſo, in the laſt article, that motion, or change of motion cannot be produced but by force impreſſed; and therefore, if we know that ſuch a cauſe exiſts, and acts upon one of the bodies, and not upon the other, we conclude that the relative motion ariſes from a change in the ſtate of reſt, or abſolute motion of the former; and that with reſpect to the latter, the effect is merely apparent. Thus, when a perſon on ſhipboard obſerves the coaſt receding from him, he is convinced that the appearance ariſes from a motion, or change of motion, in the ſhip, upon which a cauſe, ſufficient to produce this effect, acts, namely, the force of the wind or tide.

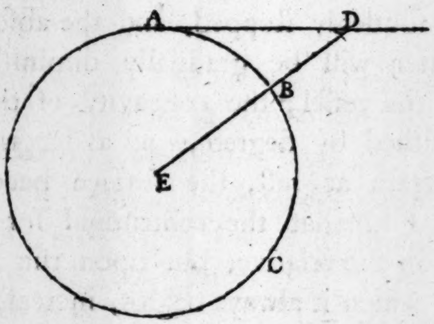
The preceſſion of the equinoxes ariſes from a real motion in the earth, and not from any motion in the heavenly bodies; becauſe we know that there is a force impreſſed upon the earth, which is ſufficient to account for the appearance.

2d. Abſolute motion may ſometimes be diſtinguiſhed from apparent motion, by the effects produced.

If a body be abſolutely in motion, it endeavours by it's inactivity to proceed in the line of it's direction; if the motion be only apparent, there is no ſuch tendency.

It is in conſequence of the tendency to perſevere in rectilinear motion that a body revolving in a circle conſtantly endeavours to recede from the center. The effort, thus produced, is called a *centrifugal force*; and as it ariſes from abſolute motion only, whenever it is obſerved, we are convinced that the motion is real.

In order to ſee the nature and origin of this force, ſuppoſe a body to deſcribe the circle *ABC*; then at any point *A*, it is moving in the direction of the tangent *AD*, and in this direction, by the firſt law of motion, it endeavours to proceed; alſo, ſince every point *D* in the tangent is without the circle, this tendency to move on in the direction of the tangent, is a tendency to recede from the center of motion; and the body will actually fly off, unleſs it is prevented by an adequate force.



The following experiment is given by **Sir I. Newton** to ſhew the effect of the centrifugal force, and to prove that it always accompanies an abſolute circular motion.

Let a bucket, partly filled with water, be ſuſpended by a ſtring, and turned round till the ſtring is conſiderably twiſted; then let the ſtring be ſuffered to untwiſt itſelf, and thus communicate a circular motion to the veſſel. At firſt the water remains at reſt, and it's ſurface is ſmooth and undiſturbed; but as it gradually acquires the motion of the bucket, the ſurface grows concave towards the center, and the water aſcends up the ſides, thus endeavouring to recede from the axis of motion; and this effect is obſerved gradually to increaſe with the abſolute velocity of the water, till at length the water and the bucket are relatively at reſt. When this is the caſe, let the bucket be ſuddenly ſtopped, and the abſolute motion of the water will be gradually diminiſhed by the friction of the veſſel; the concavity of the ſurface is alſo diminiſhed by degrees, and at length, when the water is again at reſt, the ſurface becomes plane. Thus we find that the centrifugal force does not depend upon the relative, but upon the abſolute motion, with which it always begins, increaſes, decreaſes, and diſappears.

A ſingle inſtance will be ſufficient to ſhew the great utility of this concluſion in natural philoſophy.

The diurnal rotation of the heavenly bodies may, as far as the appearance is concerned, be accounted for, either by ſuppoſing the heavens to revolve from eaſt to weſt, and complete a revolution in twenty-four hours; or, the earth to revolve from weſt to eaſt, in the ſame time: but the ſenſible diminution of gravity as we proceed towards the equator, and the oblate figure of the earth, which are the effects of a centrifugal force, prove that the appearance is to be aſcribed to a real motion in the earth.

THE

THE SECOND LAW OF MOTION.

29. *Motion, or change of motion, is proportional to the force impreſſed, and takes place in the direction in which the force acts.*

It has been ſeen in the preceding articles, that no motion or change of motion is ever produced in a body without *ſome* force impreſſed; we now aſſert that it cannot be produced without an *adequate* force; that when bodies act upon each other, the effects are not variable and accidental, but ſubject to general laws. Thus, whatever happens in one inſtance, will, under the ſame circumſtances, happen again; and when any alteration takes place in the cauſe, there will be a correſponding and proportional alteration in the effect produced. Were not cauſe and effect thus connected with, and related to, each other, we could not pretend to lay down any general rules reſpecting the mutual actions of bodies; experiment could only furniſh us with detached and iſolated facts, wholly inapplicable on other occaſions; and that harmony, which we cannot but obſerve and admire in the material world, would be loſt.

In order to underſtand the meaning and extent of this law of motion, it will be convenient to divide it into two caſes; and to point out ſuch facts, under each head, as tend to eſtabliſh it's truth.

1ſt. The ſame force, acting freely for a given time, will always produce the ſame effect, in the direction in which it acts.

Ex. 1. If a body, in one inſtance, fall perpendicularly through $16\frac{1}{12}$ feet in a ſecond, and thus acquire a velocity which would carry it, uniformly, through $32\frac{1}{6}$ feet in that time, it will always, under the ſame circumſtances, acquire the ſame velocity.

The effect produced is the ſame, whether the body begins to move from reſt or not.

Ex. 2. If a body be projected perpendicularly downwards, the velocity of projection, meaſured in feet (Art. 11), will, in one ſecond, be increaſed by $32\frac{1}{6}$; and if it be projected upwards, it will, in one ſecond, be diminiſhed by that quantity.

Ex. 3. If a body be projected obliquely, gravity will ſtill produce it's effect in a direction perpendicular to the horizon; and the body, which by it's inactivity, would have moved uniformly forward in the line of it's firſt motion, will, at the end of one ſecond, be found $16\frac{1}{12}$ feet below that line; having thus acquired a velocity of $32\frac{1}{6}$ feet per ſecond, in the direction of gravity.

2d. If the force impreſſed be increaſed or diminiſhed in any proportion, the motion communicated will be increaſed, or diminiſhed in the ſame proportion.

Ex. If a body deſcend along an inclined plane, the length of which is twice as great as it's height, the force which accelerates it's motion is half as great as the force of gravity; and, allowing for the effect of friction, and the reſiſtance of the air, the velocity generated in any time is half as great as it would have been, had the body fallen, for the ſame time, by the whole force of gravity *.

30. In eſtimating the effect of any force, two circumſtances are to be attended to; firſt, we muſt conſider what force is actually impreſſed; for this alone can produce a change in the ſtate of motion or quieſcence of a body. Thus, the effect of a ſtream upon the floats of a water-wheel is not produced by the whole force of the ſtream, but by that part of it which ariſes from the exceſs of the velocity of the water above that of the wheel; and it is nothing, if they move with equal velocities. Secondly, we muſt conſider

Same force acting freely for [illegible] time

with air

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* The experiments which moſt ſatisfactorily prove the truth of this law of motion, are made with Mr. Atwood's machine, mentioned on a former occaſion.

Let two weights, each of which is repreſented by $9m$, balance each other on this machine; and obſerve what velocity is generated, in one ſecond, when a weight $2m$ is added to either of them. Again, let the weights $8m$, $8m$, be ſuſtained, as before, and add $4m$ to one of them, then the velocity generated in one ſecond is twice as great as in the former inſtance; ſince, therefore, the maſs to be moved is the ſame in both caſes, viz. $20m$ together with the inertia of the machine, it is manifeſt that when the moving force is doubled (Art. 23), the momentum generated is alſo doubled; and, by altering the ratio of the weights, it may be ſhewn, in any other caſe, that the momentum communicated is proportional to the moving force impreſſed.

conſider in what direction the force acts; and take that part of it, only, which lies in the direction in which we are eſtimating the effect. Thus, the force of the wind actually impreſſed upon the ſails of a windmill, is not wholly employed in producing the circular motion; and therefore in calculating it's effect, in this reſpect, we muſt determine what part of the whole force acts in the direction of the motion.

In the following pages, we ſhall ſee a great variety of inſtances in which this method of eſtimating the effects of forces is applied; and the concluſions thus deduced, being found, without exception, to agree with experiment, we cannot but admit the truth of the principle.

31. Cor. Since the effect produced upon each other by two bodies, depends upon their relative velocity, it will always be the ſame whilſt this remains unaltered, whatever be their abſolute motions.

THE THIRD LAW OF MOTION.

32. *Action and reaction are equal, and in oppoſite directions.*

Matter not only perſeveres in it's ſtate of reſt or uniform rectilinear motion, but alſo by it's inertia reſiſts any change. Our experience with reſpect to this reaction, or oppoſition to the force impreſſed, is ſo conſtant and univerſal, that the very ſuppoſition of it's non-exiſtence appears to be abſurd. For who can conceive a preſſure without ſome ſupport of that preſſure? Who can ſuppoſe a weight to be raiſed without force or exertion? Thus far then we are aſſured by our ſenſes, that whenever one body acts upon another, there is *ſome* reaction: The law farther aſſerts, that the reaction is *equal in quantity* to the action.

By *action*, we here underſtand moving force, which according to the definition (Art. 22), is meaſured by the momentum which is, or would be generated, in a given time; and to determine whether action and reaction, in this ſenſe of the words, are equal or not, recourſe muſt be had to experiment.

Take two ſimilar and equal cylindrical pieces of wood, from one of which projects a ſmall ſteel point; ſuſpend them by equal ſtrings, and let one of them deſcend through any arc and impinge upon the other at reſt; then, by means of the ſteel point, the two bodies will move on together as one maſs, and with a velocity equal to half the velocity of the impinging body. Thus the momentum, which is meaſured by the quantity of matter and velocity taken jointly, remains unaltered; or, as much momentum as is gained by the body ſtruck, ſo much is taken from the momentum of the ſtriking body, or communicated to it in the oppoſite direction.

If the ſtriking body be loaded with lead, and thus made twice as heavy as the other, the common velocity after impact is found to be to the velocity of the impinging body :: 2 : 3; and becauſe the joint maſs after impact : quantity of matter in the ſtriking body :: 3 : 2, the momentum after impact : momentum before :: 3 × 2 : 2 × 3, or in a ratio of equality, as in the former caſe.

In making experiments to eſtabliſh this third law of motion, allowance muſt be made for the air's reſiſtance; and care muſt be taken to obtain a proper meaſure of the velocity before and after impact. See SIR I. NEWTON's Scholium to the Laws of Motion.

33. The third law of motion is not confined to caſes of actual impact; the effects of preſſures and attractions, in oppoſite directions, are alſo equal.

When two bodies ſuſtain each other, the preſſures in oppoſite directions muſt be equal, otherwiſe motion would enſue; and if motion be produced by the exceſs of preſſure on one ſide, the caſe coincides with that of impact *.

When one body attracts another, it is itſelf alſo equally attracted; and as much momentum as is thus communicated to one body, will alſo be communicated to the other in the oppoſite direction.

A loadſtone and a piece of iron, equal in weight, and floating upon ſimilar and equal pieces of cork, approach each other with equal velocities, and therefore with equal momenta; and when they meet, or are

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* The effects of preſſure and impact are manifeſtly of the ſame kind, and produced in the ſame way; exceſs of preſſure, on one ſide, produces momentum, and equal and oppoſite momenta ſupport each other by oppoſite preſſures.

Thus alſo preſſures may be compared, either by comparing the weights which they ſuſtain, or the momenta which they would generate under the ſame circumſtances.

are kept aſunder by any obſtacle, they ſuſtain each other by equal and oppoſite preſſures.

34. Cor. Since the action and reaction are equal at every inſtant of time, the whole effect of the action in a finite time, however it may vary, is equal to the effect of the reaction; ſince the whole effects are made up of the effects produced in every inſtant.

SCHOLIUM.

35. Theſe laws are the ſimpleſt principles to which motion can be reduced, and upon them the whole theory depends. They are not indeed ſelf-evident, nor do they admit of accurate proof by experiment, on account of the great nicety required in adjuſting the inſtruments, and making the experiments; and on account of the effects of friction, and the air's reſiſtance, which cannot entirely be removed. They are however conſtantly, and invariably, ſuggeſted to our ſenſes, and they agree with experiment as far as experiment can go; and the more accurately the experiments are made, and the greater care we take to remove all thoſe impediments which tend to render the concluſions erroneous, the more nearly do the experiments coincide with theſe laws.

Their truth is alſo eſtabliſhed upon a different ground; from theſe *general* principles innumerable *particular* concluſions have been deduced; ſometimes the deductions are ſimple and immediate, ſometimes they are made by tedious and intricate operations; yet they are all, without exception, conſiſtent with each other and with experiment: it follows therefore that the principles, upon which the calculations are founded, are true*.

36. It will be neceſſary to remember, that the laws of motion relate, *immediately*, to the actions of particles of matter upon each other, or to thoſe caſes in which the whole maſs may be conceived to be collected in a point; not to *all* the effects that may *eventually* be produced in the ſeveral particles of a ſyſtem.

A body may have a rectilinear and rotatory motion given it at the ſame time, and it will retain both. The action alſo, or reaction, may be applied at a mechanical advantage or diſadvantage, and thus they may produce, upon the whole, very different momenta; theſe effects depend upon principles which are not here conſidered; but which muſt be attended to in computing ſuch effects.

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* ATWOOD on the Motions of Bodies, p. 358.

SECTION

SECTION III.

ON THE COMPOSITION AND RESOLUTION OF MOTION.

Prop. II.

37. *TWO lines, which repreſent the momenta communicated to the ſame or equal bodies, will repreſent the ſpaces uniformly deſcribed by them in equal times; and converſely, the lines which repreſent the ſpaces uniformly deſcribed by them in equal times, will repreſent their momenta.*

The momenta of bodies may be repreſented by numbers, as was ſeen Art. 17; but in many caſes it will be much more convenient to repreſent them by lines, becauſe lines will expreſs not only the quantities of the momenta, but alſo the directions in which they are communicated.

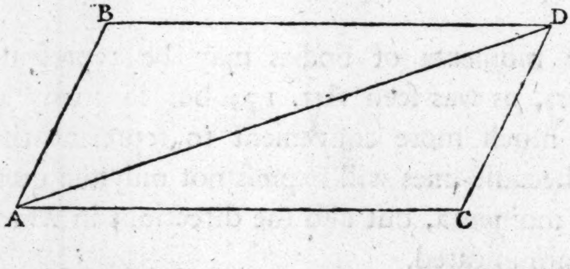
Any line drawn in the proper direction may be taken to repreſent one momentum; but to repreſent a ſecond, a line, in the direction of the latter motion, muſt be taken in the ſame proportion to the former line, that the ſecond momentum has to the firſt.

Let two lines, thus taken, repreſent the momenta communicated to the ſame, or equal bodies; then ſince $M \propto V \times \mathfrak{Q}$ (Art. 17), and $\mathfrak{Q}$ is here given, $M \propto V$; therefore the lines which repreſent the momenta, will alſo repreſent the velocities, or the ſpaces uniformly deſcribed in equal times. Again, if the lines repreſent the ſpaces uniformly deſcribed in equal times, they repreſent the velocities, and ſince $\mathfrak{Q}$ is given, $V \propto \mathfrak{Q}V \propto M$; therefore the lines repreſent the momenta.

Prop. III.

38. *Two uniform motions, which, when communicated ſeparately to a body, would cauſe it to deſcribe the adjacent ſides of a parallelogram in a given time, will, when they are communicated at the ſame inſtant, cauſe it to deſcribe the diagonal in that time; and the motion in the diagonal will be uniform.*

Let a motion be communicated to a body at *A*, which would cauſe it to move uniformly from *A* to *B* in *T*″, and at the ſame inſtant, another motion which alone would cauſe it to move uniformly from *A* to *C* in *T*″; complete the parallelogram *BC*, and draw the diagonal



diagonal *AD*; then the body will arrive at the point *D*, in *T''*, having deſcribed *AD* with an uniform motion.

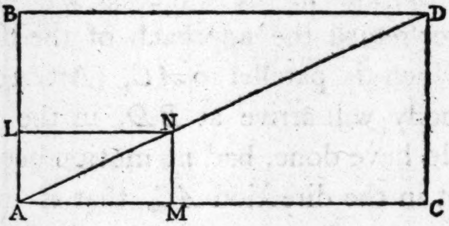
For the motion in the direction *AC* can neither accelerate nor retard the approach of the body to the line *BD* which is parallel to *AC*, (Art. 29. Ex. 3); hence the body will arrive at *BD*, in the ſame time that it would have done, had no motion been communicated to it in the direction *AC*, that is in *T''*. In the ſame manner, the motion in the direction *AB* can neither make the body approach to, nor recede from, *CD*; therefore, in conſequence of the motion in the direction *AC*, it will arrive at *CD* in the ſame time that it would have done, had no motion been communicated in the direction *AB*, that is in *T''*. Hence it follows that, in conſequence of the two motions, the body will be found both in *BD* and *CD* at the end of *T''*, and will therefore be found in *D*, the point of their interſection.

Alſo, ſince a body in motion continues to move uniformly forward in a right line, till it is acted upon by ſome external force (Art. 27), the body *A* muſt have deſcribed the right line *AD*, with an uniform motion.

39. To illuſtrate this propoſition, ſuppoſe a plane *ABDC*, as the deck of a ſhip, to be carried uniformly forward, and let the point *A* deſcribe the line *AC* in *T''*; alſo, let a body move uniformly in this plane from *A* to *B*, in the ſame time. Complete the parallelogram *BC*, and draw the diagonal *AD*. Then at the end of *T''* the body, by it's own motion, will arrive at *B*; alſo by the motion of the plane, *AB* will be brought into the ſituation *CD*, and the point *B*

will

will coincide with D; therefore the body will upon the whole, at the end of T'', be found in D. In any other time t'', let the point A be carried from A to



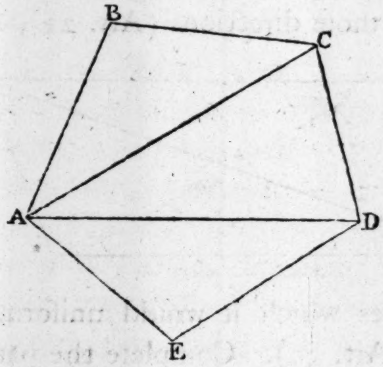
M by the motion of the plane, and the body from A to L by it's own motion; complete the parallelogram $ALNM$, and join AN; then, as in the preceding caſe, the body will, at the end of t'', be found in N; and ſince the motions in the directions AC, AB are uniform, $T : t :: AC : AM :: AB : AL$ (Art. 13); that is, the ſides of the parallelograms, about the common angle LAM, are proportional, and conſequently the parallelograms are about the ſame diagonal AD (Euc. 26.6); therefore the body at the end of any time t'' will be found in the diagonal AD. It will alſo move uniformly in the diagonal; for, from the ſimilar triangles AMN, ACD, we have $AD : AN :: AC : AM :: T : t$, or the ſpaces deſcribed are proportional to the times. (Vid. Art. 10).

40. Cor. 1. The reaſoning in the laſt Article is applicable to the motion of a point.

41. Cor. 2. If two ſides of a triangle, AB, BD, taken in order, repreſent the ſpaces over which two uniform motions would, ſeparately, carry a body in a given time; when theſe motions are communicated at the ſame inſtant to the body at A, it will deſcribe the third ſide AD, uniformly, in that time.

For, if the parallelogram *BC* be completed, the ſame motion, which would carry a body uniformly from *B* to *D*, would, if communicated at *A*, carry it in the ſame manner from *A* to *C*; and in conſequence of this motion, and of the motion in the direction *AB*, the body would uniformly deſcribe the diagonal *AD*, which is the third ſide of the triangle *ABD*.

42. Cor. 3. In the ſame manner, if the lines *AB*, *BC*, *CD*, *DE*, taken in order, repreſent the ſpaces over which any uniform motions would, ſeparately, carry a body, in a given time, theſe motions, when communicated at the ſame inſtant, will cauſe the body to deſcribe the line *AE* which completes the figure, in that time; and the motion in this line will be uniform.



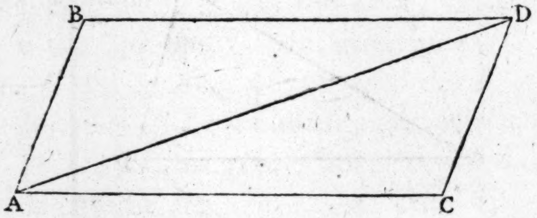
43. Cor. 4. If *AD* repreſent the uniform velocity of a body, and any parallelogram *ABDC* (Art. 38), be deſcribed about it, the velocity *AD* may be ſuppoſed to ariſe from the two uniform velocities *AB*, *AC*, or *AB*, *BD*; and if one of them, *AB*, be by any means taken away, the velocity remaining will be repreſented by *AC* or *BD*.

44. Def. A force is ſaid to be *equivalent* to any number of forces, when it will, *ſingly*, produce the ſame effect that the others produce *jointly*, in any given time.

Prop. IV.

45. *If the adjacent ſides of a parallelogram repreſent the quantities and directions of two* forces, *acting at the ſame time upon a body, the diagonal will repreſent one equivalent to them both.*

Let *AB*, *AC* repreſent two forces acting upon a body at *A*, then they repreſent the momenta communicated to it in thoſe directions (Art. 22), and conſequently the ſpaces which it would uniformly deſcribe in equal times (Art. 37). Complete the parallelogram *CB*, and draw the diagonal *AD*; then, by the laſt propoſition, *AD* is the ſpace uniformly deſcribed in the ſame time when the two motions are communicated to the body at the ſame inſtant; and ſince *AB*, *AC* and *AD*, repreſent the ſpaces uniformly deſcribed by the ſame body, in equal times, they repreſent the momenta, and therefore the forces acting in thoſe directions; that is, the forces *AB*, *AC* *, acting at the ſame time, produce a force which is repreſented, in quantity and direction, by *AD*. Def.



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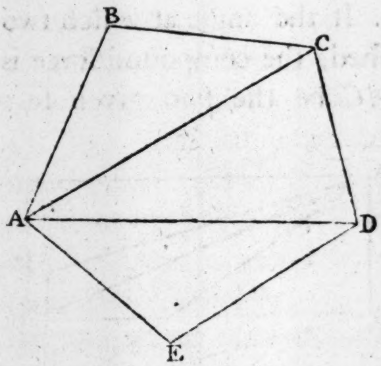
* In this, and many other caſes, where forces are repreſented by lines, the lines are uſed, for the ſake of conciſeneſs, to expreſs the forces which they repreſent.

Def. The force repreſented by *AD* is ſaid to be *compounded* of the two *AB*, *AC*.

46. Cor. 1. If two ſides of a triangle, taken in order, repreſent the quantities and directions of two forces, the third ſide will repreſent a force equivalent to them both.

For a force repreſented by *BD*, acting at *A*, will produce the ſame effect that the force *AC*, which is equal to it and in the ſame direction, will produce; and *AB*, *AC*, are equivalent to *AD*; therefore *AB*, *BD* are alſo equivalent to *AD*.

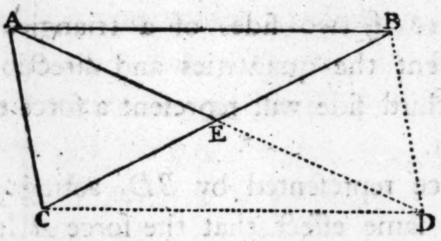
47. Cor. 2. If any lines *AB*, *BC*, *CD*, *DE*, taken in order, repreſent the quantities and directions of forces communicated at the ſame time to a body at *A*, the line *AE*, which completes the figure, will repreſent a force equivalent to them all.



For the two *AB*, *BC* are equivalent to *AC*; alſo, *AC*, *CD*, that is, *AB*, *BC*, *CD*, are equivalent to *AD*; in the ſame manner *AD*, *DE*, that is, *AB*, *BC*, *CD* and *DE*, are equivalent to *AE*.

48. Cor. 3. Let *AB* and *AC* repreſent the quantities and directions of two forces, join *BC* and draw

AE biſecting it in *E*, then will 2*AE* repreſent a force

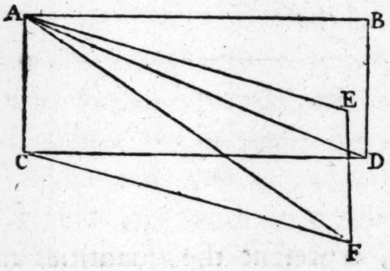


equivalent to them both.

For, if the parallelogram be completed, ſince the diagonals biſect each other, *AD*, which repreſents a force equivalent to *AB* and *AC*, is equal to 2*AE*.

49. Cor. 4. If the angle at which two given forces act be diminiſhed, the compound force is increaſed.

Let *AB*, *AC* be the two given forces, complete



the parallelogram *ABDC* and draw the diagonal *AD*, this repreſents the compound force. In the ſame manner, if *AE* be taken equal to *AB*, and *AE*, *AC*, repreſent the two forces, then *AF* the diagonal of the parallelogram *AEFC*, repreſents the compound force; and ſince the angle *BAC* is greater than the angle *EAC*, *ACD* which is the ſupplement of the former, is leſs than *ACF* the ſupplement of the latter; alſo, *CF*= *AE*

$AE = AB = CD$; therefore in the two triangles ACD, ACF, the ſides AC, CD are reſpectively equal to AC, CF, and the $\angle ACD$ is leſs than the $\angle ACF$; conſequently AD is leſs than AF (Euc. 24. 1).

50. Cor. 5. Two given forces produce the greateſt effect when they act in the ſame direction, and the leaſt when they act in oppoſite directions; for, in the former caſe, the diagonal AF becomes equal to the ſum of the ſides AC, CF; and in the latter, to their difference.

51. Cor. 6. Two forces cannot keep a body at reſt, unleſs they are equal and in oppoſite directions.

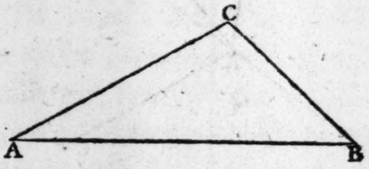
For this is the only caſe in which the diagonal, repreſenting the compound force, vaniſhes.

52. Cor. 7. In the compoſition of forces, force is loſt; for the forces repreſented by the two ſides AB, BD (Art. 45), by compoſition produce the force repreſented by AD; and the two ſides AB, BD, of a triangle, are greater than the third ſide AD.

Prop. V.

53. *If a body, at reſt, be acted upon at the ſame time by three forces which are repreſented in quantity and direction by the three ſides of a triangle, taken in order, it will remain at reſt.*

Let AB, BC, and CA, repreſent the quantities and



directions of three forces acting at the ſame time upon a body a body at *A*; then ſince *AB* and *BC* are equivalent to *AC* (Art. 46); *AB*, *BC* and *CA*, are equivalent to *AC* and *CA*; but *AC* and *CA*, which are equal and in oppoſite directions, keep the body at reſt; therefore *AB*, *BC*, and *CA*, will alſo keep the body at reſt.

Prop. VI.

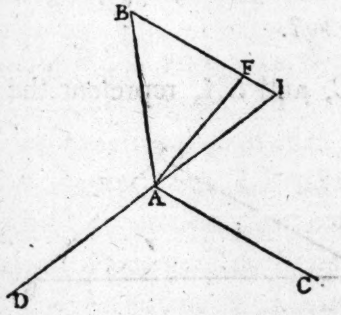
54. *If a body be kept at reſt by three forces, and two of them be repreſented in quantity and direction by two ſides* AB, BC *, *of a triangle, the third ſide, taken in order, will repreſent the quantity and direction of the other force.*

Since *AB*, *BC* repreſent the quantities and directions of two of the forces, and *AB*, *BC* are equivalent to *AC*, the third force muſt be ſuſtained by *AC*; therefore *CA* muſt repreſent the quantity and direction of the third force (Art. 51).

Prop. VII.

55. *If a body be kept at reſt by three forces, acting upon it at the ſame time, any three lines, which are in the directions of theſe forces, and form a triangle, will repreſent them.*

Let three forces acting in the directions *AB*, *AC*,



AD, keep the body *A* at reſt. In *AB* take any point *B*,

* Fig. Art. 53.

B, and through *B* draw *BI* parallel to *AC*, meeting *DA* produced in *I*; then will *AB*, *BI* and *IA* repreſent the three forces.

For *AB* being taken to repreſent the force in that direction, if *BI* do not repreſent the force in the direction *AC* or *BI*, let *BF* be taken to repreſent it; join *AF*; then ſince three forces keep the body at reſt, and *AB*, *BF* repreſent the quantities and directions of two of them, *FA* will repreſent the third (Art. 54), that is, *FA* is in the direction *AD*, which is impoſſible (Euc. 11. 1. Cor.); therefore *BI* repreſents the force in the direction *AC*; and conſequently *IA* repreſents the third force (Art. 54).

Any three lines, reſpectively parallel to *AB*, *BI*, *IA*, and forming a triangle, will be proportional to the ſides of the triangle *ABI*, and therefore proportional to the three forces.

56. Cor. 1. If three forces keep a body at reſt, they act in the ſame plane; becauſe the three ſides of a triangle are in the ſame plane (Euc. 2. 11).

57. Cor. 2. If a body be kept at reſt by three forces, any two of them are to each other inverſely as the ſines of the angles which the lines of their directions make with the direction of the third force.

Let *ABI* be a triangle whoſe ſides are in the directions of the forces; then theſe ſides repreſent the forces; and *AB* : *BI* :: ſin. *BIA* : ſin. *BAI* :: ſin. *IAC* : ſin. *BAI* :: ſin. *CAD* : ſin. *BAD*.

58. Cor. 3. If a body, at reſt, be acted upon at the ſame time by three forces, in the directions of the ſides of a triangle taken in order, and any two of them be to each other inverſely as the ſines of the angles which

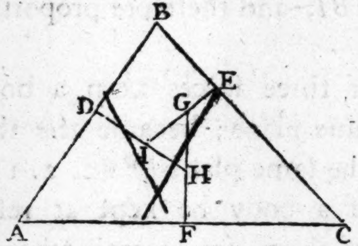
their directions make with the direction of the third, the body will remain at reſt.

For in this caſe the forces will be proportional to the three ſides of the triangle, and conſequently they will ſuſtain each other (Art. 53).

Prop. VIII.

59. *If a body be kept at reſt by three forces, and lines be drawn at right angles to the directions in which they act, forming a triangle, the ſides of this triangle will repreſent the quantities of the forces.*

Let *AB*, *BC*, *CA* be the directions in which the forces act; then the lines, *AB*, *BC*, *CA* will repreſent the forces (Art. 55). Draw the perpendiculars *DH*, *EI*, *FG*, forming a triangle *GHI*; then ſince the four angles of the quadrilateral figure *ADHF* are equal to four right angles, and the angles at *D* and *F* are right angles, the remaining angles *DHF*, *DAF* are equal to two right angles, or to the two angles *DHF*, *DHG*; conſequently, the angle *DAF* is equal to the angle *IHG*. In the ſame manner it may be ſhewn, that the angles *ABC*, *BCA* are reſpectively equal to *GIH*, *HGI*; therefore the triangles *ABC* and *GHI* are equiangular; hence, the ſides about their equal angles being proportional, the forces which are proportional



proportional to the lines *AB*, *BC*, *CA*, are proportional to the lines *HI*, *IG*, *GH*.

Cor. If the lines *DH*, *EI*, *FG* be equally inclined to the lines *DB*, *EC*, *FA*, the ſides of the triangle *GHI* will repreſent the forces.

PROP. IX.

60. *If any number of forces, repreſented in quantity and direction by the ſides of a polygon, taken in order, act at the ſame time upon a body at reſt, they will keep it at reſt.*

Let *AB*, *BC*, *CD*, *DE* and *EA* (Fig. Art. 47), repreſent the forces; then ſince *AB*, *BC*, *CD* and *DE* are equivalent to *AE* (Art. 47); *AB*, *BC*, *CD*, *DE*, and *EA*, are equivalent to *AE* and *EA*; that is, they will keep the body at reſt.

PROP. X.

61. *If any number of lines, taken in order, repreſent the quantities and directions of forces which keep a body at reſt, theſe lines will form a polygon.*

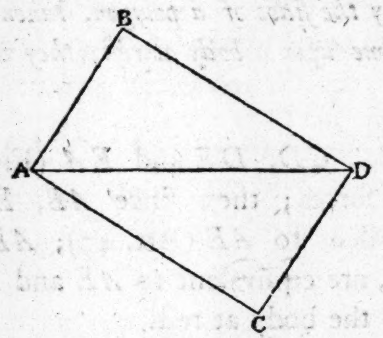
Let *AB*, *BC*, *CD* and *DE* repreſent forces which keep a body at reſt (Fig. Art. 47); then the point *E* coincides with *A*. If not, join *AE*; then *AB*, *BC*, *CD* and *DE* are equivalent to *AE*; and the body would be put in motion by a ſingle force *AE*, which is contrary to the ſuppoſition; therefore the point *E* coincides with *A*, and the lines form a polygon.

This and the laſt propoſition are true when the forces act in different planes.

Prop. XI.

62. *A ſingle force may be reſolved into any number of forces.*

Since the ſingle force *AD* is equivalent to the two *AB*, *BD*, it may be conceived to be made up of, or reſolved into, the two *AB*, *BD*. The force *AD* may therefore be reſolved into as many pairs of forces as there can be triangles deſcribed upon *AD*, or parallelograms about it. Alſo *AB*, or *BD*, may be reſolved into two; and, by proceeding in the ſame manner, the original force may be reſolved into any number of others.



63. Cor. 1. If two forces are together equivalent to *AD*, and *AB* be one of them, *BD* is the other.

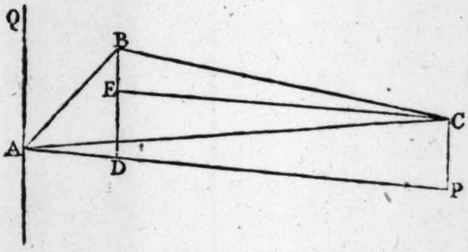
64. Cor. 2. If the force *AD* be reſolved into the two *AB*, *BD*, and *AB* be wholly loſt, or deſtroyed, the effective part of *AD* is repreſented in quantity and direction by *BD*.

65. Cor. 3. In the reſolution of forces, the whole quantity of force is increaſed. For the force repreſented by *AD* is reſolved into the two *AB*, *BD*, which are together greater than *AD* (Euc. 20. 1).

PROP. XII.

66. *The effects of forces, when eſtimated in given directions, are not altered by compoſition or reſolution.*

Let two forces *AB*, *BC*, and the force *AC* which is equivalent to them both, be eſtimated in the directions



AP, *AQ*. Draw *BD*, *CP* parallel to *AQ*; and *CE* parallel to *AP*. Then the force *AB* is equivalent to the two *AD*, *DB*; of which *AD* is in the direction *AP*, and *DB* in the direction *AQ*; in the ſame manner *BC* is equivalent to the two *BE*, *EC*; the former of which is in the direction *BD* or *QA*, and the latter in the direction *EC* or *AP*; therefore the forces *AB*, *BC*, when eſtimated in the directions *AP*, *AQ*, are equivalent to *AD*, *EC*, *DB* and *BE*; or, *AD*, *DP*, *DB* and *BE*, becauſe *EC* is equal to *DP*; and ſince *DB* and *BE* are in oppoſite directions, the part *EB* of the force *DB* is deſtroyed by *BE*; conſequently, the forces are equivalent to *AP*, *DE*, or *AP*, *PC*. Alſo *AC*, when eſtimated in the propoſed directions, is equivalent to *AP*, *PC*; therefore the effective forces in the directions *AP*, *AQ* are the ſame, whether we eſtimate *AB* and *BC*, in thoſe directions, or *AC*, which is equivalent to them.

67. Cor. When *AP* coincides with *AC*, *EC* alſo coincides with it, and *D* coincides with *E*. In this caſe the forces *DB*, *BE* wholly deſtroy each other; and thus, in the compoſition of forces, force is loſt.

SECTION IV.

ON THE MECHANICAL POWERS.

68. THE mechanical powers are the moſt ſimple inſtruments uſed for the purpoſe of ſupporting weights, or communicating motion to bodies, and by the combination of which, all machines, however complicated, are conſtructed.

Theſe powers are ſix in number, viz. the *lever*; the *wheel and axle*; the *pulley*; the *inclined* plane; the *wedge*; and the *ſcrew*.

Before we enter upon a particular deſcription of theſe inſtruments and the calculation of their effects, it is neceſſary to premiſe, that when any forces are applied to them, they are themſelves ſuppoſed to be at reſt; and conſequently, that they are either without weight, or that the parts are ſo adjuſted as to ſuſtain each other. They are alſo ſuppoſed to be perfectly ſmooth; no allowance being made for the effects of adheſion.

When two forces act upon each other by means of any machine, one of them is, for the ſake of diſtinction, called the *power*, and the other the *weight*.

ON THE LEVER.

69. Def. The *Lever* is an inflexible rod, moveable upon a point which is called *the fulcrum*, or *center of motion.*

The power and weight are ſuppoſed to act in the plane in which the lever is moveable round the fulcrum, and tend to turn it in oppoſite directions.

70. The properties of the lever cannot be deduced immediately from the propoſitions laid down in the laſt ſection, becauſe the forces acting upon the lever are not applied at a point, which is always ſuppoſed to be the caſe in the compoſition and reſolution of forces; they may however be derived from the following principles, the truth of which will readily be admitted.

Ax. 1. *If two weights balance each other upon a ſtraight lever, the preſſure upon the fulcrum is equal to the ſum of the weights, whatever be the length of the lever*.*

Ax. 2. *If a weight be ſupported upon a lever which reſts on two fulcrums, the preſſure upon the fulcrums is equal to the whole weight.*

Ax. 3. *Equal forces, acting perpendicularly at the extremities of equal arms of a lever, exert the ſame effort to turn the lever round.*

Prop.

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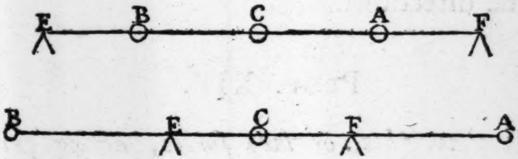
* The effect produced by the gravity of the lever is not taken into conſideration, unleſs it be expreſſly mentioned.

Prop. XIII.

71. *If two equal weights, act perpendicularly upon a ſtraight lever, the effort to put it in motion, round any fulcrum, will be the ſame as if they acted together at the middle point between them.*

Let *A* and *B* be two equal weights, acting perpendicularly upon the lever *FB*, whoſe fulcrum is *F*.



Biſect *AB* in *C*; make $CE = CF$; and at *E* ſuppoſe another fulcrum to be placed.

Then ſince the two weights *A* and *B* are ſupported by *E* and *F*, and theſe fulcrums are ſimilarly ſituated with reſpect to the weights, each ſuſtains an equal preſſure; and therefore the weight ſuſtained by *E* is equal to half the ſum of the weights. Now let the weights *A* and *B* be placed at *C*, the middle point between *A* and *B*, and conſequently the middle point between *E* and *F*; then ſince *E* and *F* ſupport the whole weight *C*, and are ſimilarly ſituated with reſpect to it, the fulcrum *E* ſupports half the weight; that is, the preſſure upon *E* is the ſame, whether the weights are placed at *A* and *B*, or collected in *C*, the middle point between them; and therefore, the effort to put the lever in motion round *F*, is the ſame on either ſuppoſition.

72. Cor. If a weight be formed into a cylinder *AB* (Fig. Art. 73) which is every where of the ſame denſity, and placed parallel to the horizon, the effort of any part *AD*, to put the whole in motion round *C*, is the ſame as if this part were collected at *E*, the middle point of *AD*.

For the weight *AD* may be ſuppoſed to conſiſt of pairs of equal weights, equally diſtant from the middle point.

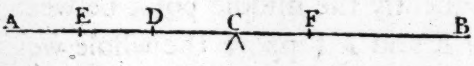
What is here affirmed of weights, is true of any forces which are proportional to the weights, and act in the ſame directions.

PROP. XIV.

73. *Two weights, or two forces, acting perpendicularly upon a ſtraight lever, will balance each other, when they are reciprocally proportional to their diſtances from the fulcrum.*

Caſe 1. When the weights act on *contrary* ſides of the fulcrum.

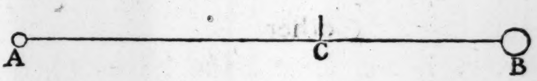
Let *x* and *y* be the two weights, and let them be formed into the cylinder *AB*, which is every where of the ſame denſity. Biſect *AB* in *C*; then this cylinder will balance itſelf upon the fulcrum *C* (Art. 72). Divide *AB* into two parts in *D*, ſo that *AD* : *DB* :: *x* : *y*, and the weights of *AD* and *DB* will be reſpectively *x* and *y*; biſect *AD* in *E* and *DB* in *F*; then ſince *AD* and *DB* keep the lever at reſt, they will keep it at reſt when



when they are collected at E and F (Art. 72); that is, x, when placed at E, will balance y, when placed at F; and $x : y :: AD : BD :: \frac{AD}{2} : \frac{BD}{2} :: \frac{AB-BD}{2} : \frac{AB-AD}{2} :: CB-BF : AC-AE :: CF : CE$.

Caſe 2. When the two forces act on the *ſame* ſide of the center of motion.

Let AB be a lever whoſe fulcrum is C, A and B two weights acting perpendicularly upon it; and let $A : B :: BC : AC$, then theſe weights will balance each other; as appears by the former caſe. Now ſuppoſe a power ſufficient to ſuſtain a weight equal to the ſum of the weights A and B, to be applied at C, in a direction oppoſite to that in which the weights act; then will this power ſupply the place of the fulcrum (Art. 70. Ax. 1); and the center of motion may be conceived to be at A or B. Let B be the center of motion; then we have a ſtraight lever whoſe center of motion is B, and the two forces A and $A+B$, acting perpendicularly upon it at the points A and C, ſuſtain each other; alſo, $A : B :: BC : AC$; therefore $A : A+B :: BC : BA$.



74. Cor. 1. If two weights, or two forces, acting perpendicularly on the arms of a ſtraight lever, keep each other in equilibrio, they are inverſely as their diſtances from the center of motion.

For the weights will balance when they are in that proportion, and if the proportion be altered by increaſing or diminiſhing one of the weights, it's effort to turn the lever round will be altered, or the equilibrium will be deſtroyed.

75. Cor. 2. Since $A : B :: BC : AC$ when there is an equilibrium upon the lever AB, whoſe fulcrum is C, by multiplying extremes and means, $A \times AC = B \times BC$.

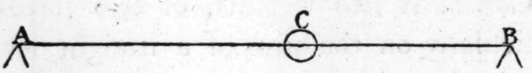
76. Cor. 3. When the power and weight act on the ſame ſide of the fulcrum, and keep each other in equilibrio, the weight ſuſtained by the fulcrum is equal to the difference between the power and the weight.

77. Cor. 4. In the common balance, the arms of the lever are equal; conſequently, the power and weight, or two weights, which ſuſtain each other, are equal. In the falſe balance, one arm is longer than the other; therefore the weight, which is ſuſpended at this arm, is proportionally leſs than the weight which it ſuſtains at the other.

78. Cor. 5. If the ſame body be weighed at the two ends of a falſe balance, it's true weight is a mean proportional between the apparent weights.

Call the true weight x, and the apparent weights, when it is ſuſpended at A and B, a and b reſpectively; then $a : x :: AC : BC$, and $x : b :: AC : BC$; therefore $a : x :: x : b$.

79. Cor. 6. If a weight C be placed upon a lever which is ſupported upon two props A and B in an

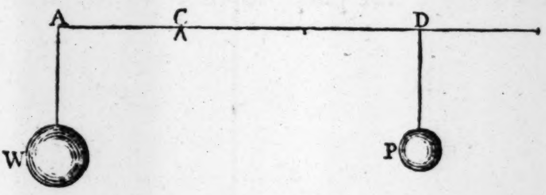


horizontal poſition, the preſſure upon A : the preſſure upon $B :: BC : AC$.

For if B be conceived to be the fulcrum, we have this proportion, the weight ſuſtained by A : the weight C ::

$C :: BC : AB$; in the ſame manner, if A be conſidered as the fulcrum, then the weight C : the weight ſuſtained by $B :: AB : CA$; therefore, ex æquo, the weight ſuſtained by A : the weight ſuſtained by $B :: BC : AC$.

80. Cor. 7. If a given weight P be moved along the graduated arm of a ſtraight lever, the weight W,



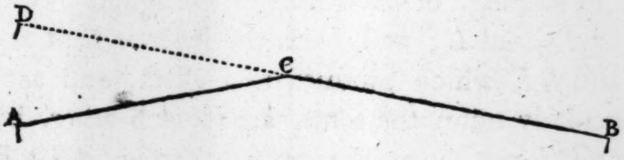
which it will balance at A, is proportional to CD, the diſtance at which the given weight acts.

When there is an equilibrium, $W \times AC = P \times DC$ (Art. 75); and AC and P are invariable; therefore $W \propto DC$ (Alg. Art. 199).

Prop. XV.

81. *If two forces, acting upon the arms of any lever, keep it at reſt, they are to each other inverſely as the perpendiculars drawn from the center of motion to the directions in which the forces act.*

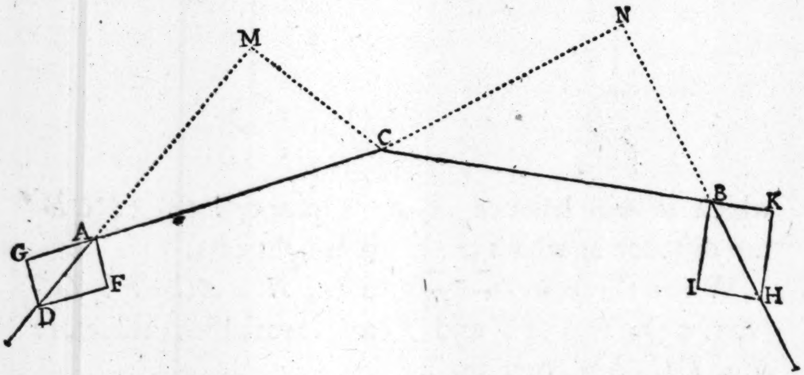
Caſe 1. Let two forces, A and B, act *perpendicularly* upon the arms CA, CB, of the lever ACB whoſe fulcrum is C, and keep each other at reſt. Produce



BC to D and make $CD = CA$; then the effort of A

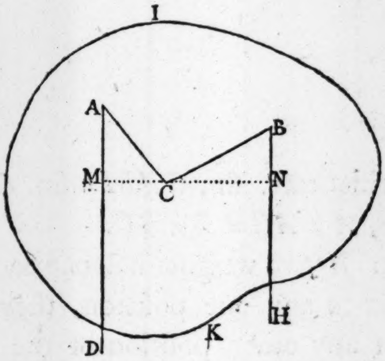
to move the lever round *C*, will be the ſame, whether it be ſuppoſed to act perpendicularly at the extremity of the arm *CA*, or *CD* (Art. 70. Ax. 3); and on the latter ſuppoſition, ſince there is an equilibrium, $A : B :: CB : CD$ (Art. 74); therefore $A : B :: CB : CA$.

Caſe 2. When the directions *AD*, *BH*, in which the forces act, are not perpendicular to the arms. Take



AD and *BH*, to repreſent the forces; draw *CM* and *CN* at right angles to thoſe directions; alſo draw *AF* perpendicular, and *DF* parallel to *AC*, and complete the parallelogram *GF*; then the force *AD* is equivalent to the two *AF*, *AG*, of which, *AG* acts in the direction of the arm, and therefore can have no effect in cauſing, or preventing any angular motion in the lever about *C*. Let *BH* be reſolved, in the ſame manner, into the two *BI*, *BK*, of which *BI* is perpendicular to, and *BK* in the direction of the arm *CB*; then *BK* will have no effect in cauſing, or preventing any angular motion in the lever about *C*; and ſince the lever is kept at reſt, *AF* and *BI*, which produce this effect, and act perpendicularly upon the arms, are to each other, by the 1ſt caſe, inverſely as the arms; that is, $AF : BI :: CB : CA$, or $AF \times CA = BI \times CB$. Alſo, in the ſimilar triangles ADF, ACM, $AF : AD :: CM : CA$, and $AF \times CA = AD \times CM$; in the ſame manner, $BI \times CB = BH \times CN$; therefore $AD \times CM = BH \times CN$, and $AD : BH :: CN : CM$.

82. Cor. 1. Let a body IK be moveable about the center C, and two forces act upon it at A and B,

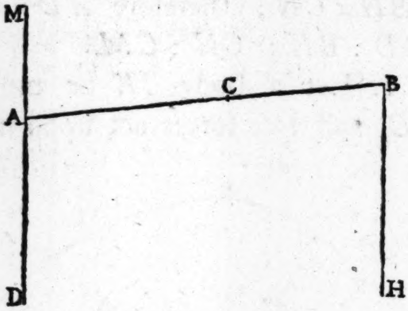


in the directions AD, BH, which coincide with the plane ACB; join AC, CB; then this body may be conſidered as a lever ACB, and drawing the perpendiculars CM, CN, there will be an equilibrium, when the force acting at A : the force acting at B :: $CN : CM$.

83. Cor. 2. The effort of the force A, to turn the lever round, is the ſame, at whatever point in the direction MD it is applied; becauſe the perpendicular CM remains the ſame.

84. Cor. 3. Since $CA : CM ::$ rad. : ſin. CAM, $CM = \frac{CA \times \text{ſin. } CAM}{\text{rad.}}$; and, in the ſame manner, $CN = \frac{CB \times \text{ſin. } CBN}{\text{rad.}}$; therefore, when there is an equilibrium, the power at A : the weight at B :: $\frac{CB \times \text{ſin. } CBN}{\text{rad.}} : \frac{CA \times \text{ſin. } CAM}{\text{rad.}} :: CB \times \text{ſin. } CBN : CA \times \text{ſin. } CAM$.

85. Cor. 4. If the lever *ACB* be ſtraight, and the directions *AD*, *BH*, parallel, $A : B :: BC : AC$;



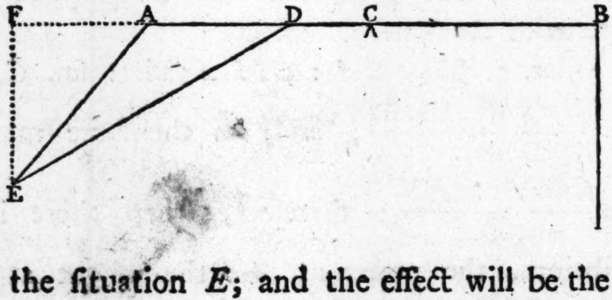
becauſe, in this caſe, ſin. $CAM =$ ſin. CBH.

Hence alſo, $A \times AC = B \times BC$.

86. Cor. 5. If two weights balance each other upon a ſtraight lever in any one poſition, they will balance each other in any other poſition of the lever; for the weights act in parallel directions, and the arms of the lever are invariable.

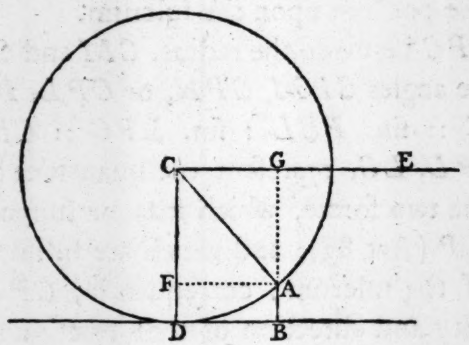
87. Cor. 6. If a man, balanced in a common pair of ſcales, preſs upwards by means of a rod, againſt any point in the beam, except that from which the ſcale is ſuſpended, he will preponderate.

Let the action upwards take place at *D*, then the ſcale, by the reaction downwards, will be brought



into the ſituation *E*; and the effect will be the ſame as if *DA*, *AE*, *DE* conſtituted one maſs; that is, drawing *EF* perpendicular to *CA* produced, as if the ſcale were applied at *F* (Art. 83); conſequently the weight, neceſſary to maintain the equilibrium, is greater than if the ſcale were ſuffered to hang freely from *A*, in the proportion of *CF* : *CA*.

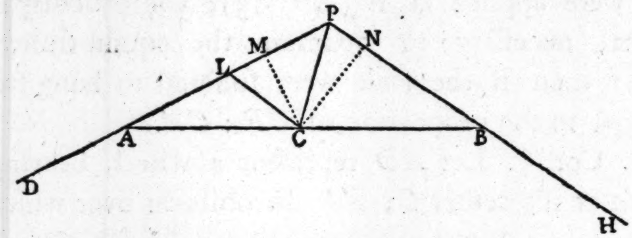
88. Cor. 7. Let *AD* repreſent a wheel, bearing a weight at it's center *C*; *AB* an obſtacle over which it is to be moved by a force acting in the direction *CE*; join *CA*, draw *CD* perpendicular to the horizon, and from *A* draw *AG*, *AF*, at right angles to *CE*, *CD*. Then *CA* may be conſidered as a lever whoſe center of motion is *A*, *CD* the direction in which the weight acts, and *CE* the direction in which the power is applied; and there is an equilibrium on this lever when the power : the weight :: *AF* :: *AG*.



Suppoſing the wheel, the weight, and the obſtacle given, the power is the leaſt when *AG* is the greateſt; that is, when *CE* is perpendicular to *CA*, or parallel to the tangent at *A*.

89. Cor. 8. Let two forces acting in the directions *AD*, *BH*, upon the arms of the lever *ACB*, keep each other in equilibrio; produce *DA* and *HB* till they meet in *P*; join *CP*, and draw *CL* parallel

to *PB*; then will *PL*, *LC* repreſent the two forces,

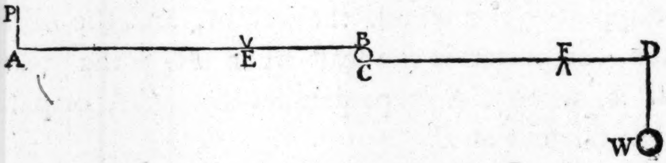


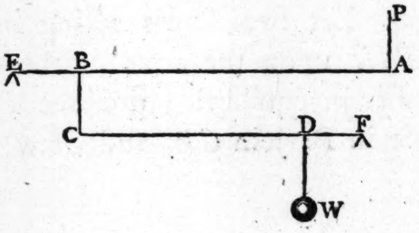
and *PC* the preſſure upon the fulcrum.

For, if *PC* be made the radius, *CM* and *CN* are the ſines of the angles *CPM*, *CPN*, or *CPL*, *PCL*; and *PL* : *LC* :: ſin. *PCL* : ſin. *LPC* :: *CN* : *CM*; therefore *PL*, *LC*, repreſent the quantities and directions of the two forces, which may be ſuppoſed to be applied at *P* (Art 83), and which are ſuſtained by the reaction of the fulcrum; conſequently, *CP* repreſents the quantity and direction of that reaction (Art. 54), or *PC* repreſents the preſſure upon the fulcrum.

Prop. XVI.

90. *In a combination of ſtraight levers*, AB, CD, *whoſe centers of motion are* E *and* F, *if they act perpendicularly upon each other, and the directions in which the power and weight are applied be alſo perpendicular to the arms, there is an equilibrium when* P : W :: EB × FD : EA × FC.





For, the power at A : the weight at B, or C :: EB : EA; and the weight at C : the weight at D :: FD : FC; therefore, $P : W :: EB \times FD : EA \times FC$.

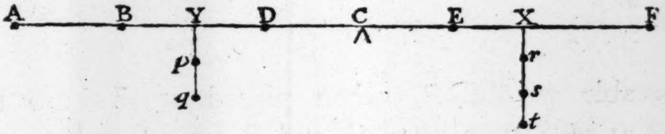
By the ſame method we may find the proportion between the power and the weight, when there is an equilibrium, in any other combination of levers.

91. Cor. If E and F be conſidered as the power and weight, A and D the centers of motion, we have, as before, $E : F :: FD \times BA : AE \times CD$. Hence the preſſure upon E : the preſſure upon F :: $FD \times BA : AE \times CD$.

Prop. XVII.

92. *Any weights will keep each other in equilibrio on the arms of a ſtraight lever, when the products, which ariſe from multiplying each weight by it's diſtance from the fulcrum, are equal, on each ſide of the fulcrum.*

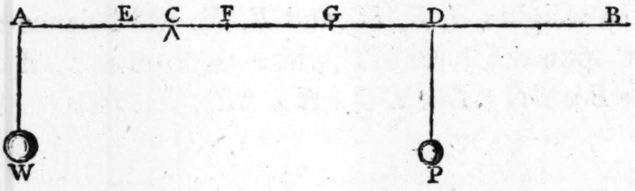
The weights A, B, D, and E, F, will balance each other upon the lever AF whoſe fulcrum is C, if $A \times AC + B \times BC + D \times DC = E \times EC + F \times FC$.



In CF take any point X, and let the weights r, s, t, placed at X, balance reſpectively, A, B, D; then $A \times AC = r \times XC$; $B \times BC = s \times XC$; $D \times DC = t \times XC$, (Art. 85); or, $A \times AC + B \times BC + D \times DC = \overline{r+s+t} \times XC$. In the ſame manner, let p and q, placed at Y, balance reſpectively, E and F; then $\overline{p+q}\times YC=E\times EC+F\times FC$; but by the ſuppoſition $A\times AC+B\times BC+D\times DC=E\times EC+F\times FC$; therefore $\overline{r+s+t}\times XC=\overline{p+q}\times YC$, and the weights r, s, t, placed at X, balance the weights p, q, placed at Y; alſo A, B, D, balance the former weights, and E, F, the latter; conſequently A, B, D, will balance E and F.

93. Cor. 1. If the weights do not act in parallel directions, inſtead of the diſtances we muſt ſubſtitute the perpendiculars, drawn from the center of motion, upon the directions. (Vid. Art. 81).

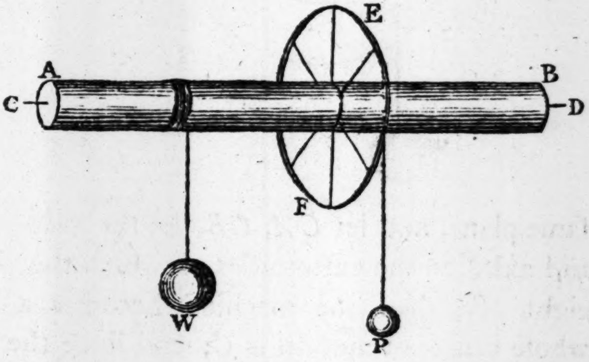
94. Cor. 2. In Art. 80 the lever is ſuppoſed to be without weight, or the arms AC, CD to balance each other: In the formation of the common *ſteel-yard* the longer arm CB is heavier than CA, and allowance muſt be made for this exceſs. Let the moveable weight P, when placed at E keep the lever at reſt; then when W and P are ſuſpended upon the lever, and the whole remains at reſt, W ſuſtains P, and alſo a weight which would ſupport P when placed at E; therefore $W\times AC=P\times DC+P\times EC=P\times DE$; and ſince AC and P are invariable, $W\propto ED$; the graduation muſt therefore begin from E; and if P, when



when placed at *F*, ſupport a weight of one pound at *A*, take *FG*, *GD*, &c. equal to each other, and to *EF*, and when *P* is placed at *G* it will ſupport two pounds; when at *D* it will ſupport three pounds, &c.

ON THE WHEEL AND AXLE.

95. The *wheel and axle* conſiſts of two parts, a cylinder *AB* moveable about it's axis *CD*, and a circle *EF* ſo attached to the cylinder that the axis *CD* paſſes through it's center, and is perpendicular to it's plane.



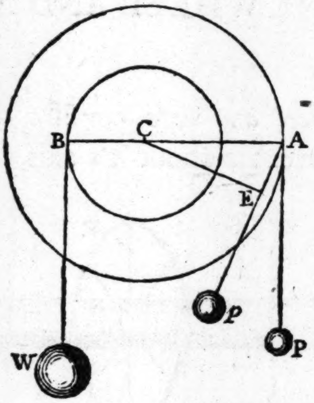
The power is applied at the circumference of the wheel, uſually in the direction of a tangent to it, and the weight is raiſed by a rope which winds round the axle in a plane at right angles to the axis.

PROP.

PROP. XVIII.

96. *There is an equilibrium upon the wheel and axle, when the power is to the weight, as the radius of the axle to the radius of the wheel.*

The effort of the power to turn the machine round the axis, muſt be the ſame at whatever point in the axle the wheel is fixed; ſuppoſe it to be removed, and placed in ſuch a ſituation that the power and weight may act in the ſame plane, and let CA, CB, be the radii of the wheel and axle, at the extremities of which the power and weight act; then the machine becomes a lever ACB, whoſe center of motion is C; and ſince the radii CA, CB, are at right angles to AP and BW, we have $P : W :: CB : CA$ (Art. 82).



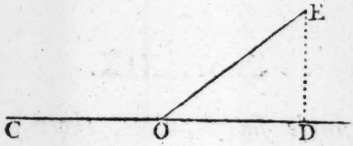
97. Cor. 1. If the power act in the direction Ap, draw CE perpendicular to Ap, and there will be an equilibrium when $P : W :: CB : CE$ (Art. 82).

The ſame concluſion may alſo be obtained by reſolving the power into two, one perpendicular to AC, and the other parallel to it.

98. Cor. 2. If $2R$ be the thickneſs of the ropes by which the power and weight act, there will be an equilibrium when $P : W :: CB+R : CA+R$, ſince the power and weight muſt be ſuppoſed to be applied in the axes of the ropes.

The ratio of the power to the weight is greater in this caſe than the former; for if any quantity be added to the terms of a ratio of leſs inequality, that ratio is increaſed (Alg. Art. 162).

99. Cor. 3. If the plane of the wheel be inclined to the axle at the angle EOD, draw ED perpendicular



to CD; and conſidering the wheel and axle as one maſs, there is an equilibrium when $P : W ::$ the radius of the axle : ED.

100. Cor. 4. In a combination of wheels and axles, where the circumference of the firſt axle is applied to the circumference of the ſecond wheel, by means of a ſtring, or by tooth and pinion, and the ſecond axle to the third wheel, &c. there is an equilibrium when $P : W ::$ the product of the radii of all the axles : the product of the radii of all the wheels. (Vid. Art. 90).

101. Cor. 5. When the power and weight act in parallel directions, and on *oppoſite* ſides of the axis, the preſſure upon the axis is equal to their *ſum*; and when they act on the *ſame* ſide, to their *difference*. In other caſes the preſſure may be eſtimated by Art. 89.

ON THE PULLEY.

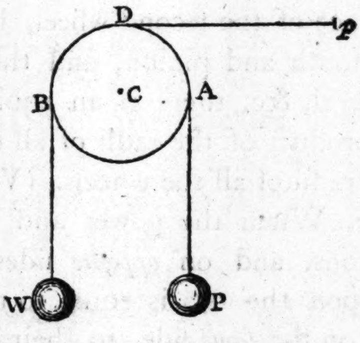
102. Def. A *Pulley* is a ſmall wheel moveable about it's center, in the circumference of which a groove is formed to admit a rope or flexible chain.

The pulley is ſaid to be *fixed*, or *moveable*, according as the center of motion is fixed or moveable.

Prop. XIX.

103. *In the ſingle fixed pulley, there is an equilibrium when the power and weight are equal.*

Let a power and weight *P*, *W*, equal to each other, act by means of a perfectly flexible rope *PDW* which paſſes over the fixed pulley *ADB*; then, whatever force is exerted at *D* in the direction *DAP*, by the power, an equal force is exerted by the weight in the direction *DBW*; theſe forces will therefore keep each other at reſt.



Cor. 1.

$P - W =$ mov^g force $\quad P + W =$ quant^t moved

$\therefore$ Acc F $= \frac{P-W}{P+W}$

$\therefore S = l F T^2 = l \times \frac{P-W}{P+W} \times T^2$

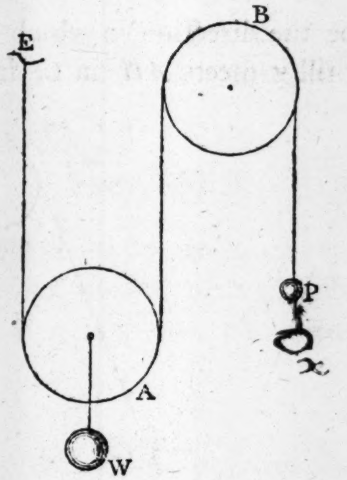
Cor. 1. Converſely, when there is an equilibrium, the power and weight are equal *.

Cor. 2. The propoſition is true in whatever direction the power is applied; the only alteration made, by changing it's direction, is in the preſſure upon the center of motion. (Vid. Art. 106).

Prop. XX.

104. *In the ſingle moveable pulley, whoſe ſtrings are parallel, the power is to the weight as 1 to 2* †.

A ſtring fixed at *E*, paſſes under the moveable pulley *A*, and over the fixed pulley *B*; the weight is annexed to the center of the pulley *A*, and the power is applied at *P*. Then ſince the ſtrings *EA*, *BA* are in



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[*] Vid. Art. 74.

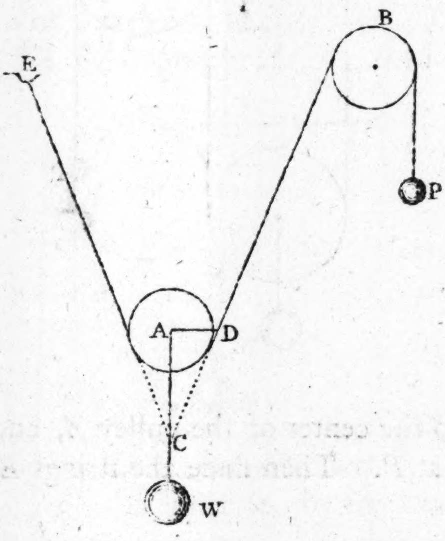
[†] In this and the following propoſitions, the power and weight are ſuppoſed to be in equilibrio.

in the direction in which the weight acts, they exactly ſuſtain it; and they are equally ſtretched in every point, therefore they ſuſtain it equally between them; or each ſuſtains half the weight. Alſo, whatever weight *AB* ſuſtains, *P* ſuſtains (Prop. XIX. Cor. 1), therefore *P* : *W* :: 1 : 2.

PROP. XXI.

105. *In general, in the ſingle moveable pulley, the power is to the weight, as radius to twice the coſine of the angle which either ſtring makes with the direction in which the weight acts.*

Let *AW* be the direction in which the weight acts; produce *BD* till it meets *AW* in *C*, from *A* draw *AD*



at right angles to *AC*, meeting *BC* in *D*; then if *CD* be taken to repreſent the power at *P*, or the power which acts in the direction *DB*, *CA* will repreſent that part of it which is effective in ſuſtaining the weight, and *AD* will be counteracted by an equal and oppoſite force, ariſing from the tenſion of the ſtring *CE*; alſo, the two ſtrings are equally effective in ſuſtaining the weight; therefore $2AC$ will repreſent the whole weight ſuſtained; conſequently, $P : W :: CD : 2AC ::$ rad. : 2 coſ. *DCA*.

106. Cor. 1. If the figure be inverted, and *E* and *B* be conſidered as a power and weight which ſuſtain each other upon the fixed pulley *A*, *W* is the preſſure upon the center of motion; conſequently, the power : the preſſure :: radius : 2 coſ. *DCA*.

107. Cor. 2. When the ſtrings are parallel, the angle *DCA* vaniſhes, and it's coſine becomes the radius; in this caſe, the power : the preſſure :: 1 : 2.

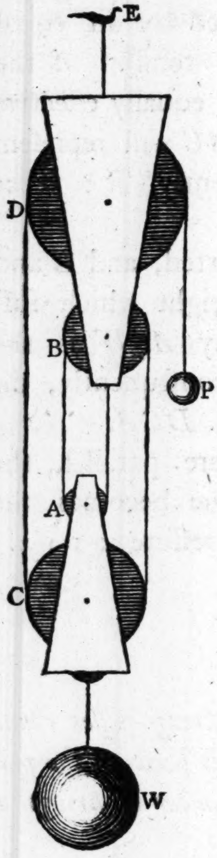
PROP. XXII.

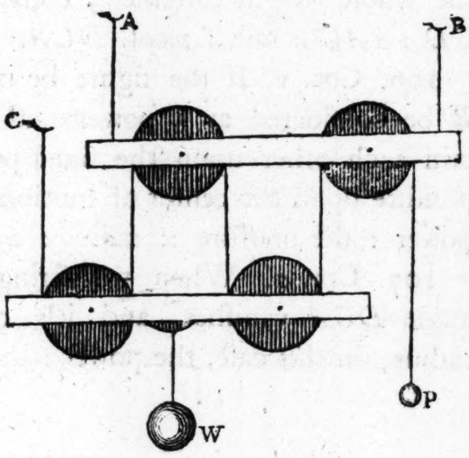
108. *In a ſyſtem where the ſame ſtring paſſes round any number of pulleys, and the parts of it between the pulleys are parallel,* P : W :: 1 : *the number of ſtrings at the lower block.*

Since the parallel parts, or ſtrings at the lower block, are in the direction in which the weight acts, they exactly ſupport the whole weight; alſo, the tenſion in every point of theſe ſtrings is the ſame, otherwiſe the ſyſtem would not be at reſt, and conſequently each of them ſuſtains an equal weight; whence

it

it follows that, if there be *n* ſtrings, each ſuſtains $\frac{1}{n}$*th*





part of the weight; therefore, *P* ſuſtains $\frac{1}{n}$*th* part of the weight, or $P : W :: \frac{1}{n} : 1 :: 1 : n$.

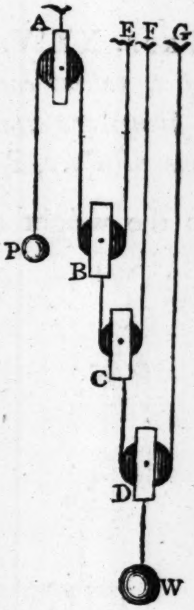
109. Cor. If two-ſyſtems of this kind be combined, in which there are *m* and *n* ſtrings, reſpectively, at the lower blocks, $P : W :: 1 : mn$.

Prop.

Prop. XXIII.

110. *In a ſyſtem where each pulley hangs by a ſeparate ſtring, and the ſtrings are parallel,* P : W :: 1 : *that power of* 2 *whoſe index is the number of moveable pulleys.*

In this ſyſtem, a ſtring paſſes over the fixed pulley *A*, and under the moveable pulley *B*, and is fixed at *E*; another ſtring is fixed at *B*, paſſes under the moveable pulley *C*, and is fixed at *F*; &c. in ſuch a manner that the ſtrings are parallel.



Then, by Art. 104, when there is an equilibrium,

P : the weight at B :: 1 : 2
the weight at B : the weight at C :: 1 : 2
the weight at C : the weight at D :: 1 : 2
&c.

Comp. $P : W :: 1 : 2 \times 2 \times 2 \times$ &c. continued to as many factors as there are moveable pulleys; that is, when there are n ſuch pulleys, $P : W :: 1 : 2^n$.

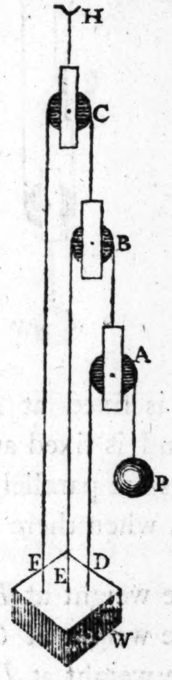
111. Cor. 1. The power and weight are wholly ſuſtained at A, E, F, G, &c. which points ſuſtain reſpectively, $2P$, P, $2P$, $4P$, &c.

112. Cor. 2. When the ſtrings are not parallel, $P : W ::$ rad. : 2 coſ. of the angle which the ſtring makes with the direction in which the weight acts, in each caſe (Art. 105).

Prop. XXIV.

113. *In a ſyſtem of* n *pulleys each hanging by a ſeparate ſtring, where the ſtrings are attached to the weight as is repreſented in the annexed figure,* $P : W :: 1 : 2^n - 1$.

A ſtring, fixed to the weight at F, paſſes over the



pulley C, and is again fixed to the pulley B; another ſtring, fixed at *E*, paſſes over the pulley *B*, and is fixed to the pulley *A*; &c. in ſuch a manner that the ſtrings are parallel:

Then, if *P* be the power, the weight ſuſtained by the ſtring *DA* is *P*; alſo the preſſure downwards upon *A*, or the weight which the ſtring *AB* ſuſtains, is $2P$ (Art. 107); therefore the ſtring *EB* ſuſtains $2P$; &c. and the whole weight ſuſtained is $P+2P+4P+$ &c. Hence, $P : W :: 1 : 1+2+4+$ &c. to *n* terms :: $1 : 2^n - 1$ (Alg. Art. 222).

114. Cor. 1. Both the power and the weight are ſuſtained at *H*.

115. Cor. 2. When the ſtrings are not parallel, the power in each caſe, is to the correſponding preſſure upon the center of the pulley :: rad. : 2 coſ. of the angle made by the ſtring with the direction in which the weight acts (Art. 106). Alſo, by the reſolution of forces, the power in each caſe, or preſſure upon the former pulley, is to the weight it ſuſtains :: rad. : coſ. of the angle made by the ſtring with the direction in which the weight acts.

ON THE INCLINED PLANE.

Prop. XXV.

116. *If a body act upon a perfectly hard and ſmooth plane, the effect produced upon the plane is in a direction perpendicular to it's ſurface.*

Caſe 1. When the body acts perpendicularly upon the plane, it's force is wholly effective in that direction; ſince there is no cauſe to prevent the effect, or to alter it's direction.

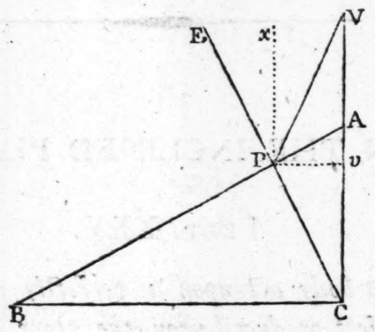
Caſe 2. When the direction in which the body acts is oblique to the plane, reſolve it's force into two, one parallel, and the other perpendicular, to the plane; the former of theſe can produce no effect upon the plane, becauſe there is nothing to oppoſe it in the direction in which it acts (Vid. Art. 29); and the latter is wholly effective (by the firſt caſe); that is, the effect produced by the force is in a direction perpendicular to the plane.

117. Cor. The reaction of the plane is in a direction perpendicular to it's ſurface (Art. 32).

Prop. XXVI.

118. *When a body is ſuſtained upon a plane which is inclined to the horizon,* P : W :: *the ſine of the plane's inclination* : *the ſine of the angle which the direction of the power makes with a perpendicular to the plane.*

Let *BC* be parallel to the horizon, *BA* a plane inclined to it; *P* a body, ſuſtained at any point upon the



the plane by a power acting in the direction PV. From P draw PC perpendicular to BA, meeting BC in C; and from C draw CV perpendicular to BC, meeting PV in V. Then the body P is kept at reſt by three forces which act upon it at the ſame time; the power, in the direction PV; gravity, in the direction VC; and the reaction of the plane, in the direction CP (Art. 117); theſe three forces are therefore properly repreſented by the three lines PV, VC and CP (Art. 55); or $P : W :: PV : VC ::$ ſin. PCV : ſin. VPC; and in the ſimilar triangles APC, ABC (Euc. 8. 6), the angles ACP, and CBA are equal; therefore $P : W ::$ ſin. ABC : ſin. VPC.

119. Cor. 1. When PV coincides with PA, or the power acts parallel to the plane, $P : W :: PA : AC :: AC : AB$.

120. Cor. 2. When PV coincides with Pv, or the power acts parallel to the baſe, $P : W :: Pv : vC :: AC : CB$; becauſe the triangles PvC, ABC are ſimilar.

121. Cor. 3. When PV is parallel to CV, the power ſuſtains the whole weight.

122. Cor. 4. Since $P : W ::$ ſin. ABC : ſin. VPC, by multiplying extremes and means, $P \times$ ſin. $VPC = W \times$ ſin. ABC; and if W, and the ſine of the $\angle ABC$ be invariable, $P \propto \frac{1}{\text{ſin.}\,VPC}$ (Alg. Art. 206); therefore P is the leaſt, when $\frac{1}{\text{ſin.}\,VPC}$ is the leaſt, or ſin. VPC the greateſt; that is, when ſin. VPC becomes the radius, or PV coincides with PA. Alſo, P is indefinitely great when ſin. VPC vaniſhes; that is, when the power acts perpendicularly to the plane.

123. Cor. 5. If P and the $\angle ABC$ be given, $W \propto$ ſin. VPC; therefore W will be the greateſt when ſin. VPC is the greateſt, that is, when PV coincides with PA. Alſo, W vaniſhes when the ſin. VPC vaniſhes, or PV coincides with PC.

124. Cor. 6. The power : the preſſure :: $PV : PC$ :: ſin. PCV : ſin. PVC :: ſin. ABC : ſin. PVC.

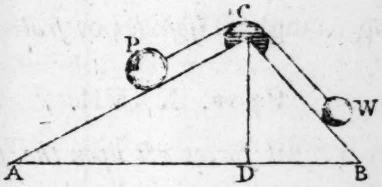
125. Cor. 7. When the power acts parallel to the plane, the power : the preſſure :: $PA : PC :: AC : BC$.

126. Cor. 8. When the power acts parallel to the baſe, the power : the preſſure :: $Pv : PC :: AC : AB$.

127. Cor. 9. $P \times$ ſin. $PVC =$ the preſſure $\times$ ſin. ABC; and when P and the $\angle ABC$ are given, the preſſure $\propto$ ſin. PVC; therefore, the preſſure will be the greateſt when PV is parallel to the baſe.

128. Cor. 10. When two ſides of a triangle, taken in order, repreſent the quantities and directions of two forces which are ſuſtained by a third, the remaining ſide, taken in the ſame order, will repreſent the quantity and direction of the third force (Art. 54). Hence, if we ſuppoſe PV to revolve round P, when it falls between Px, which is parallel to VC, and PE, the direction of gravity remaining unaltered, the direction of the reaction muſt be changed, or the body muſt be ſuppoſed to be ſuſtained againſt the under ſurface of the plane. When it falls between PE and xP produced, the direction of the power muſt be changed. And when it falls between xP produced, and PC, the directions of both the power and reaction muſt be different from what they were ſuppoſed to be in the proof of the propoſition; that is, the body muſt be ſuſtained againſt the under ſurface of the plane, by a force which acts in the direction VP.

129. Cor. 11. If the weights P, W, ſuſtain each other upon the planes AC, CB, which have a com-



mon altitude CD, by means of a ſtring PCW which paſſes over the pulley C and is parallel to the planes, then $P : W :: AC : BC$.

For, ſince the tenſion of the ſtring is every where the ſame, the ſuſtaining power, in each caſe, is the ſame; and calling this power x,

$$\begin{aligned} &P : x :: AC : CD \text{ (Art. 119)};\\ &x : W :: CD : CB;\\ \text{comp. } &P : W :: AC : CB. \end{aligned}$$

ON THE WEDGE.

130. Def. A *Wedge* is a triangular priſm; or a ſolid generated by the motion of a plane triangle parallel to itſelf, upon a ſtraight line which paſſes through one of it's angular points *.

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* See alſo Euc. B. XI. Def. 13.

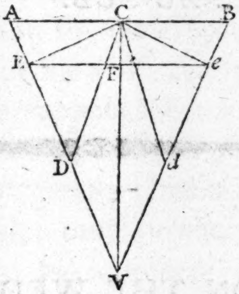
Knives, ſwords, coulters, nails, &c. are inſtruments of this kind.

The wedge is called *iſoſceles* or *ſcalene*, according as the generating triangle is *iſoſceles* or *ſcalene*.

Prop. XXVII.

131. *If two equal forces act upon the ſides of an iſoſceles wedge at equal angles of inclination, and a force act perpendicularly upon the back, they will keep the wedge at reſt, when the force upon the back is to the ſum of the forces upon the ſides, as the product of the ſine of half the vertical angle of the wedge and the ſine of the angle at which the directions of the forces are inclined to the ſides, to the ſquare of radius.*

Let AVB repreſent a ſection of the wedge, made by a plane perpendicular to it's ſides; draw VC perpendicular to AB; DC, dC, in the directions of the forces upon the ſides; and CE, Ce, at right angles to AV, BV; join Ee, meeting CV in F.



Then, in the triangles VCA, VCB, ſince the angles VCA, CAV, are reſpectively equal to VCB, VBC, and VC is common to both, $AC=CB$, and the $\angle CVA = \angle CVB$. Again, in the triangles ACD, BCd, the angles DAC, CDA, are equal to the angles CBd, BdC, and $AC=BC$; therefore, $DC=dC$. In the ſame manner it may be ſhewn that $CE=Ce$, and $AE=Be$; hence the ſides AV, BV, of the triangle AVB, are cut proportionally in E and e; therefore Ee is parallel to AB (Euc. 2. 6), or perpendicular to CV; alſo, ſince $CE=Ce$ and CF is common to the right angled triangles CEF, CeF, we have $EF=eF$ (Euc. 47. 1).

Now ſince DC and dC are equal, and in the directions of the forces upon the ſides, they will repreſent them; reſolve DC into two, DE, EC, of which DE produces no effect upon the wedge, and EC, which is effective (Art. 116), does not wholly oppoſe the power, or force upon the back; reſolve EC therefore into two, EF, parallel to the back, and FC perpendicular to it, the latter of which is the only force which oppoſes the power. In the ſame manner it appears that eF, FC are the only effective parts of dC, of which FC oppoſes the power, and eF is counteracted by the equal and oppoſite force EF; hence, if $2CF$ repreſent the power, the wedge will be kept at reſt*; that is, when the force upon the back : the ſum of the reſiſtances upon the ſides :: $2CF$: $DC+dC$ :: $2CF$: $2DC$ :: CF : DC; and

$$CF : CE :: \text{ſin.}\ CEF : \text{rad.} :: \text{ſin.}\ CVE : \text{rad.}$$
$$CE : DC :: \text{ſin.}\ CDE : \text{rad.}$$

Comp. $CF : DC :: \text{ſin.}\ CVE \times \text{ſin.}\ CDE : \overline{\text{rad.}}|^2$

132. Cor. 1. The forces do not ſuſtain *each other*, becauſe the parts DE, de are not counteracted.

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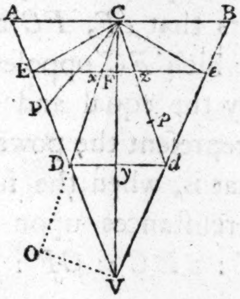
* The directions of the three forces muſt meet in a point, otherwiſe a rotatory motion will be given to the wedge.

133. Cor. 2. If the reſiſtances act perpendicularly upon the ſides of the wedge, the angle CDE becomes a right angle, and P : the ſum of the reſiſtances :: ſin. $CVE \times$ rad. : $\overline{\text{rad.}}|^2$ :: ſin. CVE : rad. :: $AC : AV$.

134. Cor. 3. If the directions of the reſiſtances be perpendicular to the back, the angle $CDE = \angle CVE$, and P : the ſum of the reſiſtances : ſin. $\overline{CVE}|^2 : \overline{\text{rad.}}|^2$ :: $AC^2 : AV^2$.

135. Cor. 4. When the reſiſtances act parallel to the back, ſin. $CDA =$ ſin. CAV, and P : the ſum of the reſiſtances :: ſin. $CVA \times$ ſin. $CAV : \overline{\text{rad.}}|^2$ :: $CA \times CV : AV^2$ :: $CE \times AV$* : AV^2 :: $CE : AV$.

136. Cor. 5. In the demonſtration of the propoſition it has been ſuppoſed that the ſides of the wedge are perfectly ſmooth; if on account of the friction, or by any other means, the reſiſtances are *wholly* effective, join Dd, which will cut CV at right angles in y, and reſolve DC, dC into Dy, yC, dy, yC, of which Dy and dy deſtroy each other, and $2yC$ ſuſtains the power. Hence, the power : the ſum of the reſiſtances :: $2yC : 2DC$ :: $yC : DC$ :: ſin. CDy or DCA : rad.



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* By ſimilar triangles, $CE : CA :: CV : AV$; therefore $CE \times AV = CA \times CV$.

137. Cor. 6. If *Ee* cut *DC* and *dC* in *x* and *z*, the forces, *xC*, *zC*, when wholly effective, and the forces *DC*, *dC*, acting upon ſmooth ſurfaces, will ſuſtain the ſame power 2*CF*.

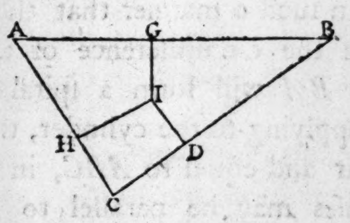
138. Cor. 7. If from any point *P* in the ſide *AV*, *PC* be drawn and the reſiſtance upon the ſide be repreſented by it, the effect upon the wedge will be the ſame as before; the only difference will be in the part *PE* which is ineffective.

139. Cor. 8. If *DC* be taken to repreſent the reſiſtance on one ſide, and *pC*, greater or leſs than *dC*, repreſent the reſiſtance on the other, the wedge cannot be kept at reſt by a power acting upon the back; becauſe, on this ſuppoſition, the forces which are parallel to the back are unequal.

This propoſition and it's corollaries have been deduced from the actual reſolution of the forces, for the purpoſe of ſhewing what parts are loſt, or deſtroyed by their oppoſition to each other; the ſame concluſions may, however, be very conciſely and eaſily obtained from Art. 142.

PROP. XXVIII.

140. *When three forces, acting perpendicularly upon the ſides of a ſcalene wedge, keep each other in equilibrio, they are proportional to thoſe ſides.*



Let *GI*, *HI*, *DI*, the directions of the forces,

meet

meet in *I*; then ſince the forces keep each other at reſt, they are proportional to the three ſides of a triangle which are reſpectively perpendicular to thoſe directions (Art. 59); that is, to the three ſides of the wedge.

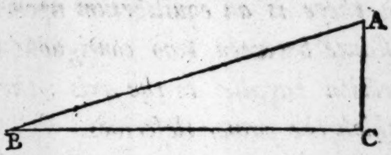
141. Cor. 1. If the lines of direction, paſſing through the points of impact, do not meet in a point, the wedge will have a rotatory motion communicated to it; and this motion will be round the center of gravity of the wedge. (Vid. Art. 184).

142. Cor. 2. When the directions of the forces are not perpendicular to the ſides, the effective parts muſt be found, and there will be an equilibrium when thoſe parts are to each other as the ſides of the wedge.

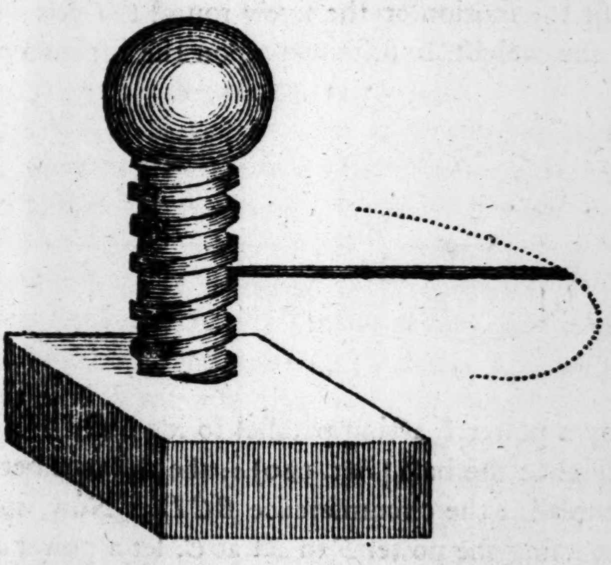
ON THE SCREW.

143. Def. The *Screw* is a mechanical power, which may be conceived to be generated in the following manner:

Let a ſolid and a hollow cylinder of equal diameters be taken, and let *ABC* be a right angled plane triangle whoſe baſe *BC* is equal to the circumference of the ſolid cylinder; apply the triangle to the convex ſurface of this cylinder in ſuch a manner that the baſe *BC* may coincide with the circumference of the baſe of the cylinder, and *BA* will form a ſpiral thread on it's ſurface. By applying to the cylinder, triangles, in ſucceſſion, ſimilar and equal to *ABC*, in ſuch a manner, that their baſes may be parallel to *BC*, the ſpiral thread may be continued; and ſuppoſing this thread to have thickneſs, or the cylinder to be protuberant where it falls, the external ſcrew will be formed, in which the



diſtance between two contiguous threads, meaſured in a direction parallel to the axis of the cylinder, is *AC*. Again, let the triangles be applied in the ſame manner to the concave ſurface of the hollow cylinder, and where the thread falls let a groove be made, and the internal ſcrew will be formed. The two ſcrews being thus exactly adapted to each other, the ſolid or hollow cylinder, as the caſe requires, may be moved round



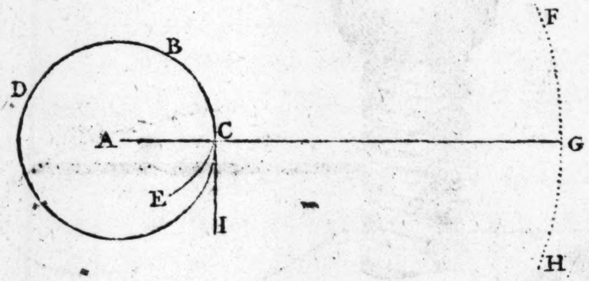
the common axis, by a lever perpendicular to that axis; and a motion will be produced in the direction of the axis, by means of the ſpiral thread.

PROP.

Prop. XXIX.

144. *When there is an equilibrium upon the ſcrew,* P : W :: *the diſtance between two contiguous threads, meaſured in a direction parallel to the axis* : *the circumference of the circle which the power deſcribes.*

Let *BCD* repreſent a ſection of the ſcrew made by a plane perpendicular to it's axis, *CE* a part of the ſpiral thread upon which the weight is ſuſtained; then *CE* is a portion of an inclined plane, whoſe height is the diſtance between two threads, and baſe equal to the circumference *BCD*. Call *F* the power which acting at *C* in the plane *BCD*, and in the direction *CI* perpendicular to *AC*, will ſuſtain the weight *W*, or prevent the motion of the ſcrew round the axis; then ſince the weight is ſuſtained upon the inclined plane *CE* by a power *F* acting parallel to it's baſe, *F* : *W* :: the height : the baſe (Art. 120) :: the diſtance between two threads : the circumference *BCD*. Now, inſtead of ſuppoſing the power *F* to act at *C*, let a power *P* act perpendicularly at *G*, on the ſtraight lever *GCA*, whoſe center of motion is *A*, and let this power produce the ſame effect at *C* that *F* does; then, by the property of the



the lever, $P : F :: CA : GA$ :: the circumference BCD : the circumference FGH. We have therefore theſe two proportions,

$$\begin{aligned} F : W &:: \text{the diſtance between two threads} : BCD \\ P : F &:: BCD : FGH \\ \text{comp. } P : W &:: \text{the diſtance between two threads} : FGH \end{aligned}$$

145. Cor. 1. In the proof of this propoſition the whole weight is ſuppoſed to be ſuſtained at one point C of the ſpiral thread; if we ſuppoſe it to be diſperſed over the whole thread, then, by the propoſition, the power at G neceſſary to ſuſtain any part of the weight : that part :: the diſtance between two threads : the circumference of the circle FGH; therefore the ſum of all theſe powers, or the whole power : the ſum of all the correſponding weights, or the whole weight, :: the diſtance between two threads : the circumference of the circle FGH (Alg. Art. 183).

146. Cor. 2. Since the power, neceſſary to ſuſtain a given weight, depends upon the diſtance between two threads and the circumference FGH, if theſe remain unaltered, the power is the ſame, whether the weight is ſuppoſed to be ſuſtained at C, or at a point upon the thread nearer to, or farther from, the axis of the cylinder.

147. Some Authors have deduced the properties of the mechanical powers *immediately* from the Third Law of Motion, contending that if the power and weight be ſuch as would ſuſtain each other, and the machine be put into motion, the momenta of the power and weight are equal; and conſequently, that the power × the velocity of the power = the weight

× the velocity of the weight; or *the power's velocity* : *the weight's velocity* :: *the weight* : *the power*.

Though this concluſion be juſt, the reaſoning by which it is attempted to be proved is inadmiſſible, becauſe the Third Law of Motion relates to the action of one body immediately upon another (Art. 36). It may however be deduced from the foregoing propoſitions; and as it is, in many caſes, the ſimpleſt method of eſtimating the power of a machine, it may not be improper to eſtabliſh it's truth.

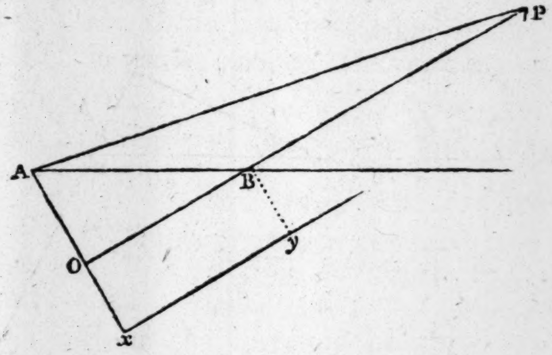
In the application of the rule, two things muſt be attended to: 1ſt. We muſt eſtimate the velocity of the power or weight in the direction in which it acts. 2dly. We muſt eſtimate that part only of the power or weight which is effective.

Theſe two conſiderations are ſuggeſted by the Second Law of Motion, according to which, motion is communicated in the *direction* of the force *impreſſed*, and is proportional to that force.

Prop. XXX.

148. *The velocity of a body in any one direction AB being given, to eſtimate it's velocity in any other direction BP.*

Suppoſe the motion of *A* to be produced by a force acting in the direction *BP*, by means of a ſtring which paſſes over a pulley at *P*; produce *PB* to *O*, making *PO* = *PA*; join *AO*; then *OB* is the ſpace which meaſures the approach of *A* to *P*. Now let the pulley be removed to ſuch a diſtance that the angle at *P* may be conſidered as evaneſcent, and the power will always act in the ſame direction BP; alſo, the angles at A and O are equal; and they are right angles, becauſe the three



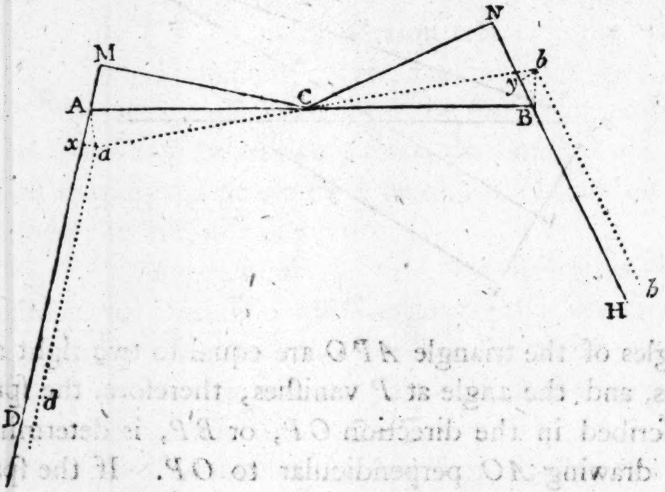
angles of the triangle APO are equal to two right angles, and the angle at P vaniſhes; therefore, the ſpace deſcribed in the direction OP, or BP, is determined by drawing AO perpendicular to OP. If the ſpace deſcribed in the direction xy, which is parallel to OP, be required, produce AO to x, and from B draw By at right angles to xy; then the figure $OByx$ is a parallelogram, and $OB=xy$ the ſpace required. Alſo, if the motion in the direction AB be uniform, the motion in the direction BP, or xy, is uniform; ſince $AB : OB ::$ rad. : coſ. ABO. Hence, the velocity in the direction AB : the velocity in the direction $BP :: AB : OB$ (Art. 11).

PROP. XXXI.

149. *If a power and weight ſuſtain each other on any machine, and the whole be put in motion, the velocity of the power : the velocity of the weight :: the weight : the power.*

Caſe

Caſe 1. *In the lever ACB*, let a power and weight, acting in the directions *AD, BH*, ſuſtain each other, and let the machine be moved uniformly round the center C, through a very ſmall angle ACa: Join Aa, Bb; draw CM, ax, at right angles to MD; and CN, by, at right angles to NB; then A's velocity : B's velocity :: $Ax : By$ (Art. 148). Now the triangles Axa, MCA are ſimilar; becauſe $\angle xAC = \angle AMC + \angle MCA$ (Euc. 32. 1), and $\angle aAC = \angle AMC$; therefore, $\angle xAa = \angle MCA$; and the angles at M and x are right angles; conſequently, the remaining angles are equal; and



$$Ax : Aa :: CM : CA;$$

alſo, in the ſim. $\triangle$s ACa, BCb, $Aa : Bb :: CA : CB$; and in the ſim. $\triangle$s Bby, BCN, $Bb : By :: CB : CN$; by compoſition, $Ax : By : CM : CN$ :: the weight : the power (Art. 81); or the power's velocity : the weight's velocity :: the weight : the power.

Caſe 2. *In the wheel and axle*, if the power be made to deſcribe a ſpace equal to the circumference of the wheel with an uniform motion, the weight will be uniformly raiſed through a ſpace equal to the circumference of the axle; hence, the power's velocity : the weight's velocity :: the circumference of the wheel : the circumference of the axle :: the radius of the wheel : the radius of the axle :: the weight : the power (Art. 96).

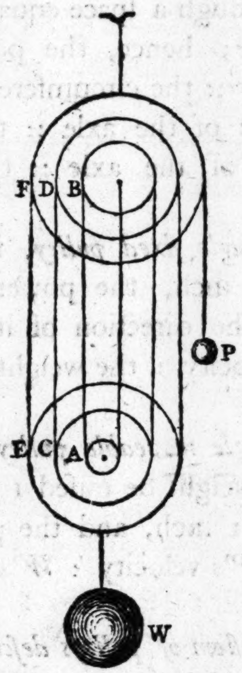
Caſe 3. *In the ſingle fixed pulley*, if the weight be uniformly raiſed 1 inch, the power will uniformly deſcribe 1 inch in the direction of it's action; therefore the power's velocity : the weight's velocity :: the weight : the power.

Caſe 4. *In the ſingle moveable pulley where the ſtrings are parallel*, if the weight be raiſed 1 inch, each of the ſtrings is ſhortened 1 inch, and the power deſcribes 2 inches; therefore, *P*'s velocity : *W*'s velocity :: *W* : *P* (Art. 104).

Caſe 5. *In the ſyſtem of pulleys deſcribed in Art.* 108, if the weight be raiſed 1 inch, each of the ſtrings at the lower block is ſhortened 1 inch, and the power deſcribes *n* inches; therefore, *P*'s velocity : *W*'s velocity :: *W* : *P*.

In this ſyſtem of pulleys, whilſt one inch of the ſtring paſſes over the pulley *A*, 2 inches paſs over the pulley *B*, 3 over *C*, 4 over *D*, &c.

Hence it follows, that if in the ſolid block *A*, the grooves *A*, *C*, *E*, &c. be cut, whoſe radii are 1, 3, 5, &c. and in the block *B*, the grooves *B*, *D*, *F*, &c. whoſe radii are 2, 4, 6, &c. and a ſtring be paſſed round theſe grooves as in the annexed figure; the grooves will anſwer the purpoſe of ſo many diſtinct pulleys, and every point in each, moving with the velocity of the ſtring in contact with it, the whole friction will be



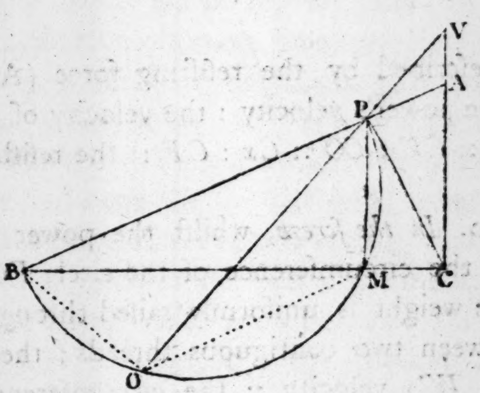
removed to the two center's of motion in the blocks *A* and *B*.

Caſe 6. *In the ſyſtem of pulleys deſcribed in Art.* 110, each ſucceeding pulley moves twice as faſt as the preceding;

therefore, *W*'s velocity : *C*'s velocity :: 1 : 2
C's velocity : *B*'s velocity :: 1 : 2
B's velocity : *P*'s velocity :: 1 : 2
&c.
comp. *W*'s velocity : *P*'s velocity :: 1 : 2 × 2 × 2 × &c. :: *P* : *W*.

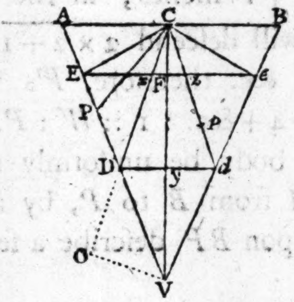
Caſe 7. *In the ſyſtem Art.* 113, if the weight be raiſed 1 inch the pulley B will deſcend 1 inch, and the pulley A will deſcend $2+1$ inches; in the ſame manner, the next pulley will deſcend $2\times\overline{2+1}+1$ inches, or $4+2+1$ inches; &c. therefore, P's velocity : W's velocity :: $1+2+4+$&c. : 1 :: $W : P$.

Caſe 8. Let a body be uniformly raiſed along *the inclined plane* BA from B to P, by a power acting parallel to PV; upon BP deſcribe a ſemicircle BOP, cutting BC in M; produce VP to O, join BO, PM, MO. Then ſince the angles BOP, BMP, in the ſemicircle, are right angles, OP and MP are ſpaces uniformly deſcribed in the ſame time, by the power and weight in their reſpective directions (Art. 148); alſo, becauſe $\angle POM=\angle PBM=\angle PCV$, and $\angle OPM=\angle PVC$ (Euc. 29. 1), the triangles POM PVC are ſimilar, and $OP : MP :: VC : PV$, or the power's velocity : the weights velocity :: the weight : the power, in the caſe of an equilibrium (Art. 118).



Caſe 9. *In the iſoſceles wedge*, xC is the only effective part of the reſiſtance DC (Vid. Art. 137); draw VO

perpendicular to CD produced; then if the wedge be moved uniformly from C to V, CO is the ſpace uni-



formly deſcribed by the reſiſting force (Art. 148); hence, the power's velocity : the velocity of the reſiſting force :: $CV : CO :: Cx : CF$ :: the reſiſtance : the power.

Caſe 10. *In the ſcrew*, whilſt the power uniformly deſcribes the circumference of the circle FGH (Art. 144), the weight is uniformly raiſed through the diſtance between two contiguous threads; therefore P's velocity : W's velocity :: the circumference of the circle FGH : the diſtance between two threads :: $W : P$.

Caſe 11. *In any combination of the mechanical powers*, let $P : W$, $W : R$, $R : S$, &c. be the ratios of the power and weight in each caſe, when there is an equilibrium; then

P's velocity : W's velocity :: $W \cdot P$
W's velocity : R's velocity :: $R : W$
R's velocity : S's velocity :: $S : R$
&c.
comp. P's velocity : S's velocity :: $S : P$

SCHOLIUM.

SCHOLIUM.

150. It has been uſual to diſtinguiſh Levers into three kinds, according to the different ſituations of the power, weight, and center of motion; there are however only two kinds which eſſentially differ; thoſe in which the forces act on *contrary* ſides of the center of motion, as the common balance, ſteelyard, &c. and thoſe in which they act on the *ſame* ſide, as the ſtock knife, ſhears which act by a ſpring, oars, &c. the proportion between the forces, when there is an equilibrium, is expreſſed in the ſame terms in each caſe; but the levers differ in this reſpect, that the preſſure upon the fulcrum depends upon the ſum of the forces in the former caſe, and upon their difference in the latter; and conſequently, the friction upon the center of motion, cæteris paribus, is greater in the former caſe than the latter.

151. The Pulley has, by ſome Writers, been referred to the lever, and they have juſtly deduced it's properties from the proportions which are found to obtain in that mechanical power; for, the adheſion of the pulley and the rope, which takes place at the circumference of the pulley, will overcome the friction at the center of motion; both becauſe it acts at a mechanical advantage, and becauſe the ſurface in contact is greater in the former caſe than in the latter; and the friction depends, not only upon the weight ſuſtained, but alſo upon the quantity of ſurface in contact:

tact: Thus, in practice, the rope and pulley move on together, and the pulley becomes a lever.

152. The Wedge has hitherto chiefly been applied to the purpoſes of ſeparating the parts of bodies, and it's power, notwithſtanding the friction, is much greater than the theory leads us to expect; the reaſon is, the effect is produced by impact, and the momentum thus generated is incomparably greater than the effect of preſſure, in the ſame time. Mr. ECKHARD, a very ingenious mechanic, by combining it with the wheel and axle, has conſtructed a machine, the power of which, conſidering it's ſimplicity, is much greater than that of any machine before invented.

SECTION V.

ON THE CENTER OF GRAVITY.

153. Def. THE *Center of Gravity* of any body, or ſyſtem of bodies, is that point upon which the body or ſyſtem, acted upon only by the force of gravity, will balance itſelf in all poſitions*.

154. Hence it follows, that if a line or plane, which paſſes through the center of gravity, be ſupported, the body, or ſyſtem, will be ſupported.

155. Converſely, if a body, or ſyſtem, balance itſelf upon a line or plane, in all poſitions, the center of gravity is in that line or plane.

If not, let the line or plane be moved parallel to itſelf till it paſſes through the center of gravity, then we have increaſed both the quantity of matter on one ſide of the line or plane, and it's diſtance from the line or plane, and diminiſhed both, on the other ſide; hence, if the body balanced itſelf in all poſitions in the former caſe, it cannot, from the nature of the lever, balance itſelf in *all poſitions*, in the latter; conſequently, the center of gravity is not in this line, or plane (Art. 154), which is contrary to the ſuppoſition.

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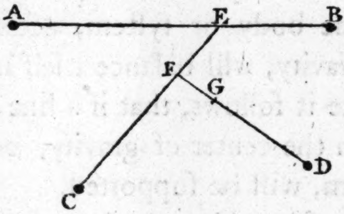
* That there is ſuch a point in every body or ſyſtem of bodies, will be ſhewn hereafter.

156. Cor. By reaſoning in the ſame manner, it appears that a body, or ſyſtem of bodies, cannot have more than one center of gravity.

Prop. XXXII.

157. *To find the center of gravity of any number of particles of matter.*

Let A, B, C, D, &c. be the particles; and ſuppoſe A, B connected by the inflexible line AB without weight *; divide AB into two parts in E, ſo that



$A : B :: BE : EA$, or comp. $A+B : B :: AB : EA$; then will A and B balance each other upon E, or if E be ſupported, A and B will be ſupported in all poſitions (Art. 86); alſo the preſſure upon the point E is equal to the ſum of the weights A and B (Art. 70. Ax. 1). Join EC, and take $A+B : C :: CF : FE$, or $A+B+C : C :: EC : FE$; then if F be ſupported, E and C will be ſupported, that is, A, B and C will be ſupported, in all poſitions of the ſyſtem; and the preſſure upon F will be the ſum of the weights, A, B and C. In the ſame manner, join FD, and divide it into two parts in G, ſo that $A+B+C : D :: DG : FG$, or $A+B+C+D : D :: FD : FG$, and the ſyſtem will balance itſelf in all poſitions upon G; that is, G is the center of gravity of the ſyſtem.

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* The particles muſt be ſuppoſed to be connected, otherwiſe they could not act upon each other, ſo as to balance upon any point.

158. Cor. 1. From this propoſition it appears that every body, or ſyſtem of bodies, has a center of gravity.

159. Cor. 2. If the particles be ſuppoſed to be connected in any other manner, the ſame point G will be found to be their center of gravity (Art. 156).

160. Cor. 3. The effect of any number of particles in a ſyſtem, to produce or deſtroy an equilibrium, is the ſame, whether they are diſperſed, or collected in their common center of gravity.

161. Cor. 4. If A, B, C, &c. be bodies of finite magnitudes, G the center of gravity of the ſyſtem, may be found by ſuppoſing the bodies collected in their reſpective centers of gravity.

162. Cor. 5. If the bodies A, B, C, &c. be increaſed or diminiſhed in a given ratio, the ſame point G will be the center of gravity of the ſyſtem. For the
E points B, F, G, depend upon the relative, and not upon the abſolute weights of the bodies.

163. Cor. 6. If any forces which are proportional to the weights, act in parallel directions at A, B, C, D, they will ſuſtain each other upon the point G; and this point is ſtill called the center of gravity, though the particles are not acted upon by the force of gravity.

164. Cor. 7. A force applied at the center of gravity of a body cannot produce a rotatory motion in

it. For every particle reſiſts, by it's inertia, the communication of motion, and in a direction oppoſite to that in which the force applied tends to communicate the motion; theſe reſiſtances, therefore, of the particles act in parallel directions, and they are proportional to the weights (Art. 25); conſequently, they will balance each other upon the center of gravity.

PROP. XXXIII.

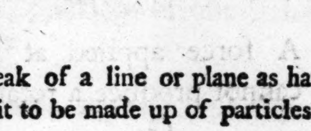
165. *To find the center of gravity of a right line**.

The center of gravity of a right line compoſed of particles of matter which are equal to each other and uniformly diſperſed, is it's middle point. For, there are equal weights on each ſide, equally diſtant from the middle point, which will ſuſtain each other, in all poſitions, upon that point (Art. 86).

PROP. XXXIV.

166. *To find the center of gravity of a parallelogram.*

Let *AB* be an uniform lamina of matter in the form of a parallelogram; biſect *AO*, *AI*, in *K* and *D*;



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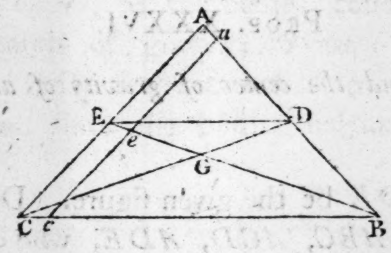
* When we ſpeak of a line or plane as having a center of gravity, we ſuppoſe it to be made up of particles of matter, uniformly diffuſed over it.

D; draw KL, DE reſpectively parallel to AI, AO, cutting each other in C; this point C is the center of gravity of the figure. For if the parallelogram be ſuppoſed to be made up of lines parallel to AI, any one of theſe, as KL, is biſected by the line DE (ſince AC, CI are parallelograms, and therefore, $KC = AD = DI = CL$); conſequently, each line will balance itſelf upon DE (Art. 165), or the whole figure will balance itſelf upon DE, in all poſitions; therefore, the center of gravity is in that line (Art. 155). In the ſame manner it may be ſhewn that the center of gravity of the figure is in the line KL, conſequently C, the interſection of the two lines, is the center of gravity required.

PROP. XXXV.

167. *To find the center of gravity of a triangle.*

Let ABC be an uniform lamina of matter in the form of a triangle; biſect AB, AC in D, E; join



CD, BE, cutting each other in G, this point is the center of gravity of the triangle.

Suppoſe the triangle to be made up of lines parallel to CA, of which let cea be one; then ſince the triangles BEC, Bec are ſimilar,

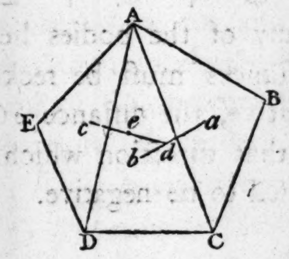
$BE : EC :: Be : ec$; alſo, in the triangles BEA, Bea, $EA : BE :: ea : Be$; by compoſition, $EA : EC :: ea : ec$; and $EA = EC$, therefore $ea = ec$; and conſequently, the line ac will balance itſelf in all poſitions upon BE. For the ſame reaſon, every other line parallel to AC will balance itſelf, in all poſitions, upon BE, or the whole triangle will balance itſelf in all poſitions upon BE; therefore the center of gravity of the triangle is in that line. In the ſame manner it may be proved that the center of gravity is in the line CD; therefore it is in G, the interſection of the two lines BE, CD.

168. Cor. The diſtance of G from B is two thirds of the line BE. Join ED; then ſince $AD = DB$, and $AE = EC$, ED is parallel to BC (Euc. 2. 6); therefore, the triangles AED, ACB are ſimilar, and $CB : CA :: ED : EA$; alternately, $CB : ED :: CA : EA :: 2 : 1$. Alſo, the triangles CGB, EGD are ſimilar, therefore, $BG : CB :: EG : ED$; alternately, $BG : GE :: CB : ED :: 2 : 1$; hence, $BG : BE :: 2 : 3$.

Prop. XXXVI.

169. *To find the center of gravity of any rectilinear figure.*

Let $ABCDE$ be the given figure. Divide it into the triangles ABC, ACD, ADE, whoſe centers of gravity a, b, c, may be found by the laſt propoſition; then if the triangles be collected in their reſpective centers of gravity (Art. 160), their common center of gravity may be found as in Prop. 32; that is, join ab and take $db : ad$ :: the triangle ABC : the triangle ADC,

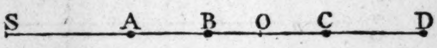


and d is the center of gravity of the two triangles ABC, ACD. Join dc, and take $ce : ed$ :: the ſum of the triangles ABC, ACD : the triangle AED, and e is the center of gravity of the figure.

Prop. XXXVII.

170. *To find the center of gravity of any number of bodies placed in a ſtraight line.*

Let A, B, C, D, be the bodies, collected in their reſpective centers of gravity; S any point in the ſtraight line SAD; O the center of gravity of all the bodies. Then ſince the bodies balance each other



upon O, $A \times AO + B \times BO = C \times CO + D \times DO$ (vid. Art. 92); that is, $A \times \overline{SO - SA} + B \times \overline{SO - SB} = C \times \overline{SC - SO} + D \times \overline{SD - SO}$; hence, by mult. and tranſpoſition, $A \times SO + B \times SO + C \times SO + D \times$

$SO = A \times SA + B \times SB + C \times SC + D \times SD$; therefore,

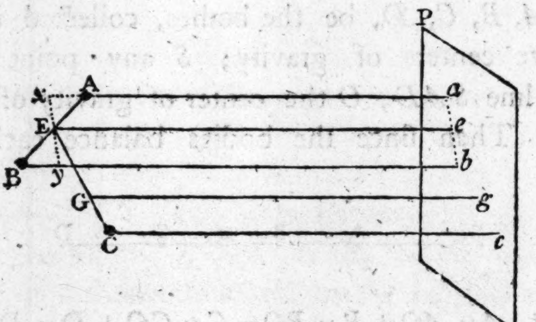
$$SO = \frac{A \times SA + B \times SB + C \times SC + D \times SD}{A + B + C + D}.$$

171. Cor. If any of the bodies lie the other way from S, their diſtances muſt be reckoned negative; and if SO be negative, the diſtance SO muſt be meaſured from S in that direction which, in the calculation, was ſuppoſed to be negative. (Vid. Alg. Art. 474).

Prop. XXXVIII.

172. *If perpendiculars be drawn from any number of bodies to a given plane, the ſum of the products of each body multiplied by it's perpendicular diſtance from the plane, is equal to the product of the ſum of all the bodies multiplied by the perpendicular diſtance of their common center of gravity from the plane.*

Let A, B, C, &c. be the bodies, collected in their reſpective centers of gravity; PQ the given plane; draw Aa, Bb, Cc at right angles to PQ, and conſequently,



conſequently, parallel to each other (Euc. 6. 11); join AB, and take $AE : EB :: B : A$, then E is the center of gravity of A and B; through E draw Ee perpendicular to $P\mathcal{Q}$, or parallel to Aa, and xEy perpendicular to Aa or Bb; then in the ſimilar triangles AEx, EBy, $Ax : AE :: By : BE$, alternately, $Ax : By :: AE : BE :: B : A$; therefore $A \times Ax = B \times By$, that is, $A \times \overline{xa - Aa} = B \times \overline{Bb - yb}$, or ſince Ea, Eb are parallelograms, $A \times \overline{Ee - Aa} = B \times \overline{Bb - Ee}$; and by multiplication and tranſpoſition, $A \times Ee + B \times Ee = A \times Aa + B \times Bb$, that is, $\overline{A+B} \times Ee = A \times Aa + B \times Bb$.

Again, join EC, and take $CG : GE :: A+B : C$, then G is the center of gravity of the bodies A, B, C; draw Gg perpendicular to $P\mathcal{Q}$; and it may be ſhewn, as before, that $\overline{A+B} \times Ee + C \times Cc = \overline{A+B+C} \times Gg$, or $A \times Aa + B \times Bb + C \times Cc = \overline{A+B+C} \times Gg$. The ſame demonſtration may be extended to any number of bodies.

173. Cor. 1. Hence $Gg = \dfrac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}$; and if a plane be drawn parallel to $P\mathcal{Q}$, and at the diſtance Gg from it, the center of gravity of the ſyſtem lies ſomewhere in this plane. In the ſame manner two other planes may be found, in each of which the center of gravity lies, and the point where the three planes cut each other is the center of gravity of the ſyſtem.

174. Cor. 2. If any of the bodies lie on the other ſide of the plane, their diſtances muſt be reckoned negative.

175. Cor. 3. Wherever the bodies are ſituated, if their reſpective perpendicular diſtances from the

plane remain the ſame, the diſtance of their common center of gravity from the plane will remain the ſame.

176. Cor. 4. Let the bodies lie in the ſame plane, and let perpendiculars, Aa, Bb, Cc, Gg be drawn to any given *line* in that plane, then

$$Gg = \frac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}.$$

177. Cor. 5. If A and B be on one ſide of the plane, and C on the other, and the plane paſs through the center of gravity, then $A \times Aa + B \times Bb = C \times Cc$. For $Gg \times \overline{A+B+C} = A \times Aa + B \times Bb - C \times Cc$, and $Gg = 0$, therefore $A \times Aa + B \times Bb - C \times Cc = 0$; or $A \times Aa + B \times Bb = C \times Cc$.

PROP. XXXIX.

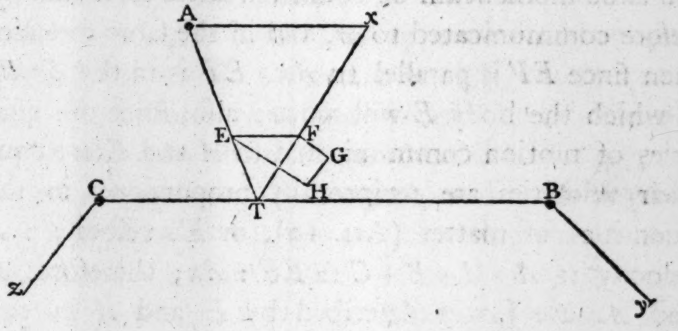
178. *If any momenta be communicated to the parts of a ſyſtem, it's center of gravity will move in the ſame manner that a body, equal to the ſum of the bodies in the ſyſtem, would move, were it placed in that center, and the ſame momenta, in the ſame directions, communicated to it.*

Let A, B, C be the bodies in the ſyſtem; join BC, and take $BT : TC :: C : B$; join AT, and take $TE : EA :: A : B+C$, or $TE : TA :: A : A+B+C$, then will E be the center of gravity of the ſyſtem (Art. 161).

Suppoſe the momentum communicated to A would cauſe it to move from A to x in T'', and at x let the body be ſtopped; join Tx, and take $TF : Tx :: A : A+B+C$, then F is the center of gravity of the bodies when they are at x, B, C; join EF, and ſince $TE : TA$

:: A

$:: A : A+B+C :: TF : Tx$, EF is parallel to Ax (Euc. 2. 6), and conſequently the triangles TEF, TAx are ſimilar; therefore $EF : Ax :: A : A+B+C$.



Hence if one body A in the ſyſtem be moved from A to x, the center of gravity is moved from E to F; which point may be thus determined; draw EF parallel to Ax, and take $EF : Ax :: A : A+B+C$.

Next let a momentum be communicated to B which would cauſe it to move from B to y in T''; at y let the body be ſtopped; then, according to the rule above laid down, draw FG parallel to By, and take $FG : By :: B : A+B+C$, and G will be the center of gravity of the bodies when they are at x, y, C. In the ſame manner, let a momentum be communicated to C which would cauſe it to move from C to z in T'', and at z let the body be ſtopped; draw GH parallel to Cz and take $GH : Cz :: C : A+B+C$, then H is the center of gravity of the bodies when they are at x, y, z. If now the momenta, inſtead of being communicated ſeparately, be communicated *at the ſame inſtant* to the bodies, at the end of T'' they will be found in *x*, *y*, *z* reſpectively; therefore at the end of *T″* their common center of gravity will be in *H*.

Now let *E* be a body equal to *A*+*B*+*C*, and let the ſame momentum be communicated to it that was before communicated to *A*, and in the ſame direction; then ſince *EF* is parallel to *Ax*, *EF* is in the *direction* in which the body *E* will move; alſo, ſince the quantities of motion communicated to *A* and *E* are equal, their velocities are reciprocally proportional to their quantities of matter (Art. 19), or *E*'s velocity : *A*'s velocity :: *A* : *A*+*B*+*C* :: *EF* : *Ax*; therefore, *EF* and *Ax* are ſpaces deſcribed by *E* and *A* in equal times (Art. 11), or *E* will deſcribe the ſpace *EF* in *T″*. In the ſame manner *FG* is the ſpace which the body *E* will deſcribe in *T″*, if the momentum, before communicated to *B*, be communicated to it; and *GH* the ſpace it will deſcribe in *T″*, if the momentum before communicated to *C*, be communicated to it; join *EH*; and when the motions are communicated *at the ſame inſtant* to *E*, it will deſcribe *EH* in *T″* (Art. 42). Hence it follows, that when the ſame momenta are communicated to the parts of a ſyſtem, and to a body, equal to the ſum of the bodies, placed in the common center of gravity, this body and the center of gravity are in the ſame point at the end of *T″*; and *T* may repreſent any time; therefore, they are always in the ſame point.

The ſame demonſtration may be applied whatever be the number of bodies in the ſyſtem.

179. Cor. 1. If the parts of a ſyſtem move uniformly in right lines, the center of gravity will either remain at reſt, or move uniformly in a right line. For if the momenta communicated to the ſeveral parts of

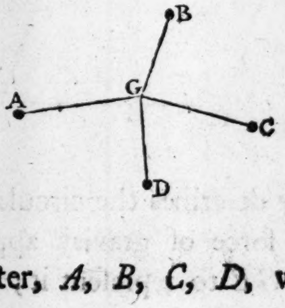
the ſyſtem be communicated to a body, equal to the ſum of the bodies, placed in the center of gravity of the ſyſtem, it will either remain at reſt or move uniformly in a right line (Art. 27).

180. Cor. 2. If two weights ſupport each other upon any machine, and it be put in motion, the center of gravity of the weights will neither aſcend nor deſcend. For the momenta of the weights in a direction perpendicular to the horizon, are equal and oppoſite (Art. 149); therefore, if they were communicated to a body equal to the ſum of the bodies, placed in the common center of gravity, they would neither cauſe it to aſcend or deſcend.

181. Cor. 3. The motion or quieſcence of the center of gravity is not affected by the mutual actions of the parts of a ſyſtem upon each other. For action and reaction are equal and in oppoſite directions, and equal and oppoſite momenta communicated to a body, equal to the ſum of the bodies in the ſyſtem, will not diſturb it's motion or quieſcence.

182. Cor. 4. The effect of any force to communicate motion to the common center of gravity, is the ſame, upon whatever body in the ſyſtem it acts.

183. Cor. 5. If *G* be the center of gravity of the particles of matter, *A*, *B*, *C*, *D*, which are acted upon



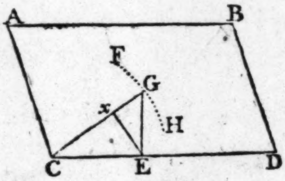
upon only by their mutual attractions, they will meet at *G*. For they muſt meet, and their common center of gravity will remain at reſt (Art. 181); therefore, they muſt meet at that center.

184. Cor. 6. If a rotatory motion be communicated to a body and it be then left to move freely, the axis of rotation will paſs through the center of gravity. For the center of gravity itſelf, either remaining at reſt or moving uniformly forward in a right line, has no rotation.

PROP. XL.

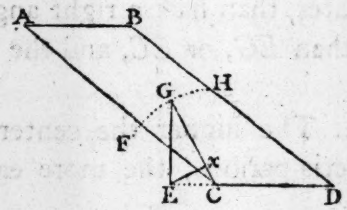
185. *If a body be placed upon an horizontal plane, and a line be drawn from it's center of gravity perpendicular to that plane, the body will be ſuſtained, or not, according as the perpendicular falls within or without it's baſe.*

Let *ABDC* repreſent the body, *G* it's center of gravity; draw *GE* perpendicular to the horizon; join *CG*, and with the radius *CG* deſcribe the circular arc *HGF*; then the body cannot fall over at *C* unleſs the center of gravity deſcribes the circular arc *GF*. Suppoſe the whole force of gravity applied at *G* (Art. 160), and take *GE* to repreſent it; draw *Ex* perpendicular



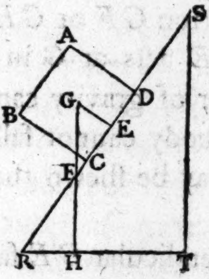
dicular to CG; then the force GE is equivalent to the two Gx, xE, of which Gx cannot move the body either in the direction GF or GH; and when E falls within the baſe, xE acts at G in the direction GH; therefore the center of gravity cannot deſcribe the arc GF, that is, the body cannot fall over at C. In the ſame manner it may be ſhewn that it cannot fall over at D.

When the perpendicular GE falls without the baſe, xE acts in the direction GF, and ſince there is no force to counteract this, the center of gravity will move in that direction, or the body will fall.



186. Cor. 1. In the ſame manner it may be ſhewn, that if a body be placed upon an inclined plane, and the lateral motion be prevented by friction, the body will be ſuſtained or not, according as the perpendicular to the horizon, drawn through it's center of gravity, falls within or without the baſe.

Ex. Let $ABCD$ repreſent a cube of uniform denſity placed upon the inclined plane RS; G it's center of gravity; draw GE perpendicular to CD, and GFH perpendicular to the horizon; then this body will not be ſuſtained upon the inclined plane, if the angle of the plane's inclination SRT, exceed half a right angle. For if the $\angle FRH$ be greater than half a right angle, the



$\angle RFH$, or GFE, is leſs than half a right angle, and the $\angle FGE$ is greater than half a right angle; therefore, EF is greater than EG, or EC, and the body will roll.

187. Cor. 2. The higher the center of gravity of a body is, cæteris paribus, the more eaſily it is overturned.

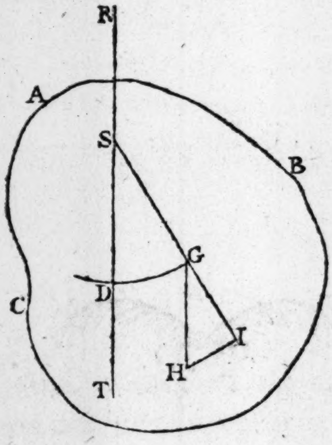
The ſame conſtruction being made as in the propoſition, the whole weight of the body : that part of the weight which keeps it ſteady upon it's baſe, or oppoſes any power employed to overturn it :: $GE : xE :: GC : CE$; and when CE and the whole weight of the body are given, the force which keeps the body ſteady $\propto \frac{1}{GC}$ (Alg. Art. 206); therefore as GC increaſes, that is, as GE increaſes, the force which keeps the body ſteady decreaſes, or the more eaſily will the body be overturned.

188. Cor. 3. When CE vaniſhes with reſpect to GC, the force which keeps the body ſteady vaniſhes, and the body may be overturned by a very ſmall force. Thus it is extremely difficult to balance a body upon a *point* placed *under* the center of gravity.

Prop. XLI.

189. *If a body be ſuſpended by any point, it will not remain at reſt till the center of gravity is in the line which is drawn through that point, perpendicular to the horizon.*

Let S be the point of ſuſpenſion of the body ABC;



G it's center of gravity; join SG and produce it; through S, and G, draw RST, and GH perpendiculars to the horizon; then the *immediate* effect of gravity is to draw the point G in the direction GH; take GH to repreſent the force in that direction, and draw HI perpendicular to GI; then the force GH is equivalent to the two GI, IH, of which GI is ſuſtained by the reaction of the point of ſuſpenſion S, and IH is employed in moving the center of gravity in a direction perpendicular to SG; therefore the center of gravity cannot remain at reſt till IH vaniſhes; that is, till the angle *IGH*, or *GST* vaniſhes, or *SG* coincides with *RT*.

190. Cor. Hence it follows that if a body be ſuſpended ſucceſſively by different points, and perpendiculars to the horizon be drawn through the points of ſuſpenſion, the center of gravity will lie in each of theſe perpendiculars, and conſequently, in the point of their interſection.

* * *

SECTION VI.

ON THE COLLISION OF BODIES.

191. Def. HARDNESS, which is found in different bodies in different degrees, conſiſts in a firm coheſion of the component particles; and that body is ſaid to be harder than another, whoſe particles require a greater force to ſeparate them. By a *perfectly hard* body we mean one whoſe parts cannot be ſeparated, or moved one amongſt another by any finite force.

192. Def. The tendency in a body to recover it's former figure, after having been compreſſed, is called *elaſticity*. That body is ſaid to be more elaſtic than another, which recovers it's figure with the greater force, ſuppoſing the compreſſing force the ſame. By a *perfectly elaſtic* body we mean one which recovers it's figure with a force equal to that which was employed in compreſſing it.

That ſuch a tendency exiſts in bodies is evident from a variety of experiments. If an ivory ball, ſtained with ink, be brought gently into contact with an unſtained ball, the ſpot received by the latter will be very ſmall, ſince two ſpheres touch each other only in a ſingle point; but if one of the balls be made to impinge upon the other, the ſpot will be enlarged; and the greater the force of impact, the greater will be the ſurface ſtained; hence it is manifeſt, that one, or both of the balls, has been compreſſed, and afterwards recovered it's ſpherical figure. Almoſt all bodies with which we are acquainted are elaſtic in a greater or leſs degree; but none perfectly ſo. In ſteel balls the force of elaſticity is to the compreſſing force as 5 to 9; in glaſs as 15 to 16; though in all caſes, the force of elaſticity ſeems to depend, in ſome meaſure, upon the diameter of the ball.

193. Def. The impact of two bodies is ſaid to be *direct*, when their centers of gravity move in the right line which paſſes through the point of impact.

In conſidering the effects of colliſion, the bodies are uſually ſuppoſed to be ſpheres of uniform denſity; and in their actions upon each other, not to be affected by gravity, or any other force but that of inertia.

Prop. XLII.

194. *If the impact of two perfectly hard bodies be direct, after impact they will either remain at reſt, or move on uniformly, together.*

Since there is no force to turn either body out of the line of direction, they will continue in that line after impact.

impact*. Let *A* and *B* be the two bodies, moving in the *ſame* direction, and let *A* overtake *B*; then will *A* continue to accelerate *B*'s motion, and *B* will continue to retard *A*'s, till their velocities are equal, at which time they will ceaſe to act upon each other; and ſince there is no force to ſeparate them, they will move on together, and their common velocity, by the Firſt Law of Motion, will be uniform. When they move in *oppoſite* directions, if their forces be equal they will reſt after impact; if *A*'s force be greater than *B*'s, the whole velocity of *B* will be deſtroyed, and *A*'s not being deſtroyed, *A* will communicate velocity to *B*, and *B* by it's reaction will retard *A*, till they move on together, as in the former caſe.

PROP. XLIII.

195. *If the impact of two perfectly hard bodies be direct, their common velocity may be found by dividing the whole momentum before impact, eſtimated in the direction of either motion, by the ſum of the quantities of matter.*

Let *A* and *B* be the quantities of matter contained in the bodies, *a* and *b* their velocities; then, when they move in the ſame direction, $Aa+Bb$ is the whole momentum in that direction, before impact. When they move in oppoſite directions, $Aa - Bb$ is the whole momentum eſtimated in the direction in which *A* moves.

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* The momenta of the particles in each body are proportional to their weights, ſince their velocities are equal; theſe momenta, therefore, will not turn the body to either ſide of the line paſſing through the center of gravity (Art. 163).

In the former caſe, as much as Aa, the momentum of A, is diminiſhed, ſo much is Bb, the momentum of B, increaſed by the impact (Art. 32); therefore $Aa + Bb$ is equal to the whole momentum after impact.

In the latter caſe, if Aa be greater than Bb, before the bodies can begin to move together, Bb, the momentum of B, muſt be deſtroyed; and therefore A's momentum muſt be diminiſhed by the quantity Bb (Art. 32). Thus, when the bodies begin to move in the ſame direction, $Aa - Bb$ is their whole momentum; and as much momentum as is afterwards communicated to B, ſo much is loſt by A; therefore $Aa - Bb$ is equal to the whole momentum after impact.

If Aa be leſs than Bb, the momentum after impact, in the direction of B's motion, will be $Bb - Aa$; or, in the direction of A's motion, $Aa - Bb$.

Let v be the common velocity after impact; then $\overline{A+B} \times v$ is the whole momentum; conſequently, $\overline{A+B} \times v = Aa \pm Bb$, and $v = \frac{Aa \pm Bb}{A+B}$. In which expreſſion, the poſitive ſign is to be uſed when the bodies move in the *ſame* direction before impact, and the negative ſign when they move in *oppoſite* directions.

196. Cor. 1. When the bodies move in oppoſite directions with equal momenta, they will remain at reſt after impact. In this caſe $Aa - Bb = 0$; therefore $v = 0$.

197. Cor. 2. If Bb be greater than Aa, v is negative. This ſhews that the bodies will move in the direction of B's motion, which was ſuppoſed, in the propoſition, to be negative.

PROP. XLIV.

198. *In the direct impact of two perfectly hard bodies* A *and* B, *eſtimating the effects in the direction of* A's *motion,*

motion, A+B : A :: *the relative velocity of the two bodies* : *the velocity gained by* B. *And* A+B : B :: *their relative velocity* : *the velocity loſt by* A.

The ſame notation being retained; when the bodies move in the ſame direction, $a-b$ is their relative velocity (Art. 12); and v, their common velocity after impact, is $\frac{Aa+Bb}{A+B}$ (Art. 195); therefore, the velocity *gained* by B, or $v-b$, is $\frac{Aa+Bb}{A+B}-b$, or $\frac{Aa-Ab}{A+B}$; hence, $A+B : A :: a-b$: the velocity gained by B. Alſo, $a-\frac{Aa+Bb}{A+B}$, or $\frac{Ba-Bb}{A+B}$ is the velocity *loſt* by A; therefore $A+B : B :: a-b$: the velocity loſt by A.

When the bodies move in oppoſite directions, $a+b$ is their relative velocity (Art. 12); and $v=\frac{Aa-Bb}{A+B}$ (Art. 195); alſo, the velocity communicated to B upon the whole, in the direction of A's motion, is $b+v$, or $b+\frac{Aa-Bb}{A+B}$; that is, $\frac{Aa+Ab}{A+B}$; therefore, $A+B : A :: a+b$: the velocity gained by B.

The velocity loſt by A is $a-\frac{Aa-Bb}{A+B}$, or $\frac{Ba+Bb}{A+B}$; therefore, $A+B : B :: a+b$: the velocity loſt by A.

Ex. Let the weights of A and B be 10 and 6*; their velocities 12 and 8, reſpectively; then, when they move in the ſame direction, $10+6 : 10 :: 12-8 : \frac{40}{16}=2\frac{1}{2}$, the velocity gained by B; and $10+6 : 6 :: 12-8 : \frac{24}{16}=1\frac{1}{2}$, the velocity loſt by A.

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* Vid. Art. 26.

When they move in oppoſite directions, $12+8$ is their relative velocity; and $10+6 : 10 :: 12+8 : \frac{200}{16}=12\frac{1}{2}$, the velocity gained by B in the direction of A's motion. Alſo, ſince it had a velocity 8 in the oppoſite direction before impact, it's velocity after impact is $4\frac{1}{2}$ in the direction of A's motion. Again, $10+6 : 6 :: 12+8 : \frac{120}{16}=7\frac{1}{2}$, the velocity loſt by A.

199. Cor. 1. Whilſt the relative velocity remains the ſame, the velocity gained by B, and the velocity loſt by A, are unaltered.

200. Cor. 2. Hence it alſo follows that the velocities, gained by B, and loſt by A, are the ſame, whether both bodies are in motion, or A impinges upon B at reſt, with a velocity equal to their relative velocity in the former caſe.

201. Cor. 3. If the relative velocity be the ſame, the momentum communicated is the ſame, whether A impinges upon B, or B upon A.

Call r the relative velocity; then when A impinges upon B, $A+B : A :: r : \frac{Ar}{A+B}$, the velocity gained by B; therefore $\frac{ABr}{A+B}$ is the momentum gained by B. When B impinges upon A, $A+B : B :: r : \frac{Br}{A+B}$, the velocity gained by *A*; therefore $\frac{ABr}{A+B}$ is the momentum gained by *A*; which is alſo the momentum gained by *B* on the former ſuppoſition.

Prop. XLV.

202. *When the bodies are perfectly elaſtic, the velocity gained by the body ſtruck, and the velocity loſt by the ſtriking body, will be twice as great as if the bodies were perfectly hard.*

Let *A* and *B* be the bodies; then, as in Art. 194, *A* will accelerate *B*'s motion, and *B* will retard *A*'s, till their velocities are equal; and if they were perfectly hard they would then ceaſe to act upon each other, and move on together; thus, during the firſt part of the colliſion, the ſame effect is produced, that is, the ſame velocity is gained and loſt, as if the bodies were perfectly hard. But during this period the bodies are compreſſed by the ſtroke, and ſince they are, by the ſuppoſition, perfectly elaſtic, the force with which each will recover it's former ſhape is equal to that with which it was compreſſed; therefore, each body will receive another impulſe from the elaſticity equal to the former, or *B* will gain, and *A* loſe upon the whole, twice as great a velocity as if both bodies had been perfectly hard.

203. The ſame demonſtration may be applied to the caſe where one body is perfectly hard, and the other perfectly elaſtic.

Prop. XLVI.

204. *In the colliſion of two perfectly elaſtic bodies* A *and* B, A+B : 2A :: *their relative velocity before impact* : *the velocity gained by* B *in the direction of* A*'s motion*; *and* A+B : 2B :: *their relative velocity* : *the velocity loſt by* A, *in that direction.*

Call r the relative velocity of the bodies, x the velocity gained by B, and y the velocity loſt by A, when both bodies are perfectly hard; then $2x$ is the velocity gained by B, and $2y$ the velocity loſt by A, when they are perfectly elaſtic; and

$A+B : A :: r : x$ (Art. 198); therefore,
$A+B : 2A :: r : 2x$ (Alg. Art. 185), the velocity gained by B.

Again, $A+B : B :: r : y$ (Art. 198); therefore,
$A+B : 2B :: r : 2y$, the velocity loſt by A.

Ex. Let the weights of the bodies be 5 and 4, their velocities 7 and 5; then, when they move in the ſame direction, $5+4 : 10 :: 7-5 : \frac{20}{9} = 2\frac{2}{9}$, the velocity gained by B; therefore $5+2\frac{2}{9}$, or $7\frac{2}{9}$ is B's velocity after impact. Alſo, $5+4 : 8 :: 7-5 : \frac{16}{9} = 1\frac{7}{9}$, the velocity loſt by A; therefore $7-1\frac{7}{9}$, or $5\frac{2}{9}$, is A's velocity after impact. When they move in oppoſite directions, $5+4 : 10 :: 7+5 : \frac{120}{9} = 13\frac{1}{3}$, the velocity gained by *B*. Alſo, ſince it had a velocity 5 in the oppoſite direction, it's velocity after impact, in the direction of *A's* motion, is $13\frac{1}{3} - 5$, or $8\frac{1}{3}$. Again, $5+4 : 8 :: 7+5 : \frac{96}{9} = 10\frac{2}{3}$ *A*'s velocity loſt; and ſince it had a velocity 7 before impact, after impact it will move in the oppoſite direction with a velocity $3\frac{2}{3}$.

205. Cor. 1. When $A = B$, the bodies interchange velocities. For, in this caſe, $A + B = 2A = 2B$; therefore, the velocity gained by *B*, and the velocity loſt by *A*, are reſpectively equal to their relative velocity before impact. Let *a* and *b* be their velocities before impact; then, when they move in the ſame direction, $a - b$ is the velocity gained by *B*, or loſt by *A*; therefore $a - b + b$, or *a*, is *B*'s velocity after impact; and $a - \overline{a - b}$, or *b*, is *A*'s velocity. If *b* be negative, or the bodies move in oppoſite directions, $a + b - b$, or *a*, is *B*'s velocity, and $a - \overline{a + b}$, or $-b$, is *A*'s velocity after impact.

206. Cor. 2. If the bodies move in oppoſite directions with equal quantities of motion, the whole momentum of each will be deſtroyed during the compreſſion, and an equal one generated by elaſticity in the oppoſite direction; each body will therefore be reflected with a velocity equal to that which it had before impact.

207. Cor. 3. In the congreſs of perfectly elaſtic bodies, the relative velocity after impact is equal to the relative velocity before impact.

Let a and b be the velocities of the bodies before impact; p and q their velocities after; then $a-b=q-p$. For, $A+B : 2A :: a-b : \frac{2A\times\overline{a-b}}{A+B}$, the velocity gained by B; therefore $q=b+\frac{2A\times\overline{a-b}}{A+B}$. Alſo, $A+B : 2B :: a-b : \frac{2B\times\overline{a-b}}{A+B}$, the velocity loſt by A; therefore $p=a-\frac{2B\times\overline{a-b}}{A+B}$; and $q-p=b-a+\frac{\overline{2A+2B}\times\overline{a-b}}{A+B}$ $=b-a+2a-2b=a-b$. When the bodies A and B move in oppoſite directions, the ſign of b is negative; in other reſpects the demonſtration is the ſame.

208. Cor. 4. The ſum of the products of each body multiplied by the ſquare of it's velocity, is the ſame before and after impact.

The notation in the laſt article being retained; $Aa+Bb=Ap+Bq$ (Art. 34); by tranſpoſition, $Aa-Ap=Bq-Bb$; or $A\times\overline{a-p}=B\times\overline{q-b}$. Alſo $a-b=q-p$ (Art. 207); or $a+p=q+b$; therefore $A\times\overline{a-p}\times\overline{a+p}=B\times\overline{q-b}\times\overline{q+b}$; or $Aa^2-Ap^2=Bq^2-Bb^2$; therefore $Aa^2+Bb^2=Ap^2+Bq^2$. If any of the quantities, b, p, q, be negative, it's ſquare will be poſitive, and therefore the concluſion will not be altered.

209. Cor. 5. If there be a row of equal elaſtic bodies, A, B, C, D, &c. at reſt, and a motion be communicated to A, and thence to B, C, D, &c. they will all remain at reſt after the impact, except the laſt, which will move off with a velocity equal to that with which the firſt moved.

For A and B will interchange velocities (Art. 205); that is, A will remain at reſt, and B move on with A's velocity. In the ſame manner it may be ſhewn that all the others will remain at reſt after impact, except the laſt, which will move off with the velocity communicated to *A*.

210. Cor. 6. If the bodies decreaſe in magnitude, they will all move in the direction of the firſt motion, and the velocity communicated to each ſucceeding body will be greater than that which was communicated to the preceding.

For, $A+B : 2B ::$ *A*'s velocity before impact : the velocity loſt by *A*; and ſince $2B$ is leſs than $A+B$, *A* does not loſe it's whole velocity; therefore it will move on after impact in the direction of the firſt motion. Alſo, $A+B : 2A ::$ *A*'s velocity before impact : the velocity gained by *B*; and ſince $2A$ is greater than $A+B$, the velocity gained by *B* is greater than *A*'s velocity before impact. In the ſame manner it may be ſhewn that *B*, *C*, *D*, &c. will move on in the direction of the firſt motion; and that the velocity communicated to each will be greater than that which was communicated to the preceding body.

211. Cor. 7. If the bodies increaſe in magnitude they will all be reflected back, except the laſt, and the velocity communicated to each ſucceeding body will be leſs than that which was communicated to the preceding.

For, in this caſe, $2B$ is greater than $A+B$; therefore, *A* loſes more than it's whole velocity, or it will move in the contrary direction. Alſo, $2A$ is leſs than $A+B$; therefore, the velocity gained by *B* is leſs than *A*'s velocity before impact. In the ſame manner it may be ſhewn that *B*, *C*, *D*, &c. will be reflected;

and that the velocity communicated to each will be leſs than that which was communicated to the preceding body.

212. Cor. 8. The velocity thus communicated from A through B to C, when B is greater than one of the two A, C, and leſs than the other, exceeds the velocity which would be communicated immediately from A to C.

Let a repreſent A's velocity; then

$A+B : 2A :: a : \frac{2Aa}{A+B}$, the velocity of B; and $B+C : 2B :: \frac{2Aa}{A+B} : \frac{2Aa}{A+B} \times \frac{2B}{B+C}$, the velocity communicated from B to C.

Again, $A+C : 2A :: a : \frac{2Aa}{A+C}$, the velocity communicated immediately from A to C. Hence it follows, that the velocity communicated to C, by means of B, is greater than that which would be communicated to it immediately, if $\frac{2Aa}{A+B} \times \frac{2B}{B+C}$ be greater than $\frac{2Aa}{A+C}$; that is, if $A+C$ be greater than $\frac{\overline{A+B} \times \overline{B+C}}{2B}$, or $2A+2C$ greater than $A+C+B+\frac{AC}{B}$; or $A+C$ greater than $B+\frac{AC}{B}$. Suppoſe $A=B+x$, $C=B+y$; then $A+C=2B+x+y$, and $B+\frac{AC}{B}=B+\frac{B^2+Bx+By+xy}{B}=2B+x+y+\frac{xy}{B}$; therefore, the velocity communicated to C by means of B, is greater than the velocity communicated to it without B, if $2B+x+y$ be greater than $2B+x+y+\frac{xy}{B}$, which will always be the caſe when xy is negative, or when x and y have different ſigns; that is, when B is leſs than one of the bodies, A, C, and greater than the other *.

213. Cor. 9. If the bodies be in geometrical progreſſion, the velocities communicated to them will be in geometrical progreſſion; and when there are n ſuch bodies, whoſe common ratio is r, the velocity of the firſt : the velocity of the laſt $:: \overline{1+r}|^{n-1} : 2^{n-1}$.

Let A, Ar, Ar^2, Ar^3, &c. be the bodies; a, b, c, d, &c. the velocities ſucceſſively communicated to them; then

$A+Ar : 2A :: a : b$, or
$1+r : 2 :: a : b$; and in the ſame manner,
$1+r : 2 :: b : c$
$1+r : 2 :: c : d$ &c.

therefore $a : b :: b : c :: c : d$ &c. Alſo, by compoſition, $\overline{1+r}|^{n-1} : 2^{n-1} :: a$: the velocity of the laſt.

214. Cor. 10. If the number of mean proportionals, interpoſed between two given bodies A and X, be increaſed without limit, the ratio of A's velocity to the velocity thus communicated to X will approximate to the ratio of $\sqrt{X} : \sqrt{A}$ as it's limit.

Let A, B, C, D, ... X be the bodies; a, b, c d, ... x the velocities communicated to them. Then

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* The velocity communicated from A through B to C, is a maximum when A, B, and C are in geometrical progreſſion (Flux. Art. 21. Ex. 9).

Then ſince the number of bodies interpoſed between A and X is increaſed without limit, their differences will be diminiſhed without limit; let $A+z=B$; then

$$2A+z : 2A :: a : b$$

$$\text{or } A+\frac{z}{2} : A :: a : b$$

$$^{*}\text{and } A+\frac{z}{2} : A :: \sqrt{A+z} : \sqrt{A} :: \sqrt{B} : \sqrt{A};$$

$$\text{therefore, } \sqrt{B} : \sqrt{A} :: a : b$$

$$\text{in the ſame manner, } \sqrt{C} : \sqrt{B} :: b : c$$

$$\sqrt{D} : \sqrt{C} :: c : d$$

&c.

$$\text{comp. } \sqrt{X} : \sqrt{A} :: a : x.$$

Prop. XLVII.

215. *In the direct impact of two imperfectly elaſtic bodies* A *and* B, *if the compreſſing force be to the force of elaſticity* :: 1 : m, *then* A+B : $\overline{1+m}\times$ A :: *their relative velocity before impact : the velocity gained by* B *in the direction of* A*'s motion. And* A+B : $\overline{1+m}\times$ B :: *their relative velocity before impact : the velocity loſt by* A, *in that direction.*

By reaſoning as in Art. 202, it appears that the velocity gained by *B*, and the velocity loſt by *A*, during the compreſſion, are the ſame as if the bodies were perfectly hard; and the velocity communicated by the elaſticity is to the velocity communicated by the compreſſion $:: m : 1$. Call r the relative velocity before impact, x the velocity gained by B, and y the velocity loſt by A, during the compreſſion; then $\overline{1+m}\times x$ is the velocity gained by B, and $\overline{1+m}\times y$ the velocity loſt by A, upon the whole. Now

---

* Since $\overline{A+\frac{z}{2}}\big|^2 : A^2 :: A^2+Az+\frac{z^2}{4} : A^2 :: A+z+\frac{z^2}{4A} : A :: B+\frac{z^2}{4A} : A$, the ratio of $\overline{A+\frac{z}{2}}\big|^2 : A^2$, when z is continually diminiſhed, approximates to the ratio of $B : A$, and conſequently, the ratio of $A+\frac{z}{2} : A$ approximates to the ratio of $\sqrt{B} : \sqrt{A}$ as it's limit.

$$A+B : A :: r : x \text{ (Art. 198)},$$

and $A+B : B :: r : y$;

therefore, $A+B : \overline{1+m}\times A :: r : \overline{1+m}\times x$, the velocity gained by B;

and $A+B : \overline{1+m}\times B :: r : \overline{1+m}\times y$, the velocity loſt by A.

216. Cor. 1. The relative velocity before impact : the relative velocity after impact $:: 1 : m$.

Let a and b be the velocities of the two bodies before impact, p and q their velocities after; then

$A+B : \overline{1+m}\times A :: a-b : \dfrac{\overline{1+m}\times A\times\overline{a-b}}{A+B}$, the velocity gained by B;

therefore, $q=b+\dfrac{\overline{1+m}\times A\times\overline{a-b}}{A+B}$;

in the ſame manner, $p=a-\dfrac{\overline{1+m}\times B\times\overline{a-b}}{A+B}$;

hence, $q-p=b-a+\dfrac{\overline{1+m}\times\overline{A+B}\times\overline{a-b}}{A+B}$, or

$b-a+a-b+m\times\overline{a-b}$, i. e. $m\times\overline{a-b}$, is the relative velocity after impact; and $a-b : m\times\overline{a-b} :: 1 : m$.

When the bodies move in oppoſite directions the ſign of b is negative.

217. Cor. 2. Hence it appears that if the velocities of the bodies before and after impact be known, the elaſtic force is known.

218. Cor. 3. If A impinge upon B at reſt, A will remain at reſt after impact when $A : B :: m : 1$.

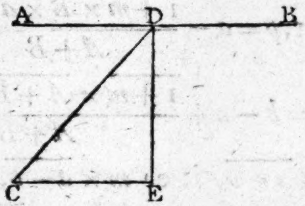
In this caſe A loſes it's whole velocity, and $A+B : \overline{1+m}\times B :: a :$ the velocity loſt by A; therefore $A+B=\overline{1+m}\times B$, and $A=mB$; conſequently, $A : B :: m : 1$.

219. Cor. 4. The momentum communicated is the ſame, whether A impinges upon B, or B upon A, if the relative velocity be the ſame. This is the caſe when the bodies are perfectly hard (Art. 201); and the effect produced in elaſtic bodies is in a given ratio to that which is produced when the bodies are perfectly hard.

Prop. XLVIII.

220. *When a perfectly hard body impinges obliquely on a perfectly hard and immoveable plane* AB, *in the direction* CD, *after impact it will move along the plane, and the velocity before impact* : *the velocity after* :: *radius* : *the coſine of the angle* CDA.

Take CD to repreſent the motion of the body before



impact; draw CE parallel, and DE perpendicular to AB.

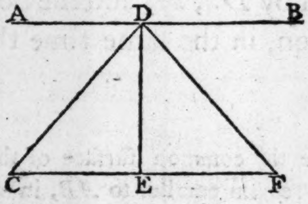
Then *CD* may be reſolved into the two *CE*, *ED* (Art. 43), of which *ED* is wholly employed in carrying the body in a direction perpendicular to the plane; and ſince the plane is immoveable, this motion will be wholly deſtroyed, (Vid. Art. 116). The other motion *CE*, which is employed in carrying the body parallel to the plane, will not be affected by the impact; and conſequently, there being no force to ſeparate the body and the plane, the body will move along the plane; and it will deſcribe $DB = CE$ in the ſame time that it deſcribed *CD* before impact; alſo, theſe ſpaces are uniformly deſcribed (Art. 27); conſequently, the velocity before impact : the velocity after :: $CD : CE$ :: radius : ſin. $\angle CDE$ :: radius : coſ. $\angle CDA$.

221. Cor. The velocity before impact : the difference between the velocity before and the velocity after, that is, the velocity loſt :: radius : rad. – coſ. $\angle CDA$ :: rad. : the verſed ſine of the angle *CDA*.

Prop. XLIX.

222. *If a perfectly elaſtic body impinge upon a perfectly hard, or perfectly elaſtic, immoveable plane* AB, *in the direction* CD, *it will be reflected from it in the direction* DF, *which makes, with* DB, *the angle* BDF *equal to the angle* ADC.

Let *CD* repreſent the motion of the impinging



body; draw *CF* parallel, and *DE* perpendicular to

AB; make $EF=CE$, and join *DF*. Then the whole motion may be reſolved into the two *CE*, *ED*, of which *CE* is employed in carrying the body parallel to the plane, and muſt therefore remain after the impact*; and *ED* carries the body in the direction *ED*, perpendicular to the plane; and ſince the plane is immoveable, this motion will be deſtroyed during the compreſſion, and an equal motion will be generated in the oppoſite direction by the force of elaſticity. Hence it appears, that the body at the point *D* has two motions, one of which would carry it uniformly from *D* to *E*, and the other from *E* to *F*, in the ſame time, viz. in the time in which it deſcribed *CD* before the impact; it will, therefore, deſcribe *DF* in that time (Art. 38). Alſo, in the triangles *CDE*, *EDF*, *CE* is equal to *EF*, the ſide *ED* is common, and the $\angle CED$ is equal to the $\angle DEF$; therefore, the $\angle CDE=$ the $\angle EDF$; hence, the $\angle CDA=$ the $\angle FDB$.

223. Cor. 1. Since $CD=DF$, and theſe are ſpaces uniformly deſcribed in equal times, before and after the impact, the velocity of the body after reflection is equal to it's velocity before incidence.

224. Cor. 2. If the body and plane be imperfectly elaſtic, take $DE : Dx ::$ the force of compreſſion : the force of elaſticity; draw *xf* parallel and equal to *EF*, join *Ff*, *Df*; then the two motions which the body has at *D* are repreſented by *Dx*, *xf*, and the body will deſcribe *Df*, after reflection, in the ſame time that it deſcribed *CD*

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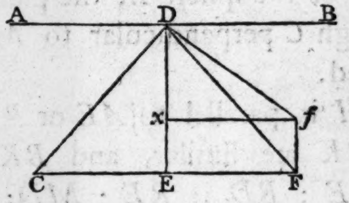
* Here we ſuppoſe the common ſurface of the body and plane, during the impact, to remain parallel to *AB*, in which caſe there is no cauſe to accelerate or retard the motion *CE* (Vid. Art. 116).

CD before incidence; therefore, the velocity before

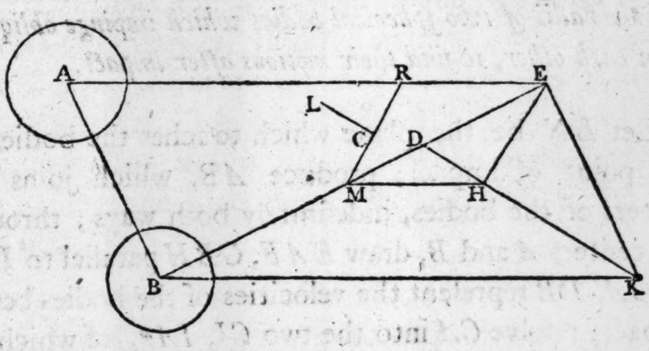


incidence : the velocity after reflection :: $CD : Df$ :: $DF : Df$ :: ſin. DfF, or ſin. of it's ſupplement EDf : ſin. DFf, or ſin. FDE :: ſin. EDf : ſin. EDC.

Prop. L.

225. *Having given the radii of two ſpherical bodies, their velocities, and the directions in which they move, to find the plane which touches them both at the point of impact.*

Let AE, BE be the directions in which the bodies A and B move; and let AE and BD be ſpaces uniformly deſcribed by them in the ſame time; complete the parallelogram $ABKE$; join KD, and with the center E and radius equal to the ſum of the radii of the two bodies, deſcribe a circular arc cutting KD in H;



join EH, and complete the parallelogram $EHMR$. Then R and M will be the places of the centers of the two ſpheres when they meet; and if RC be taken equal to the radius of the ſphere A, the plane CL, which is drawn through C perpendicular to MR, will be the plane required.

Since MH is parallel to AE or BK, the triangles DMH, DBK are ſimilar, and $BK : BD :: MH : MD$; or $AE : BD :: RE : MD$; therefore $AE : BD :: AR : BM$ (Euc. 19. 5); and ſince AE and BD are ſpaces deſcribed in the ſame time by the uniform motions of A and B, AR and BM, which are proportional to them, will be deſcribed in the ſame time; when, therefore, the center of the body A is in R, the center of the body B is in M, and the diſtance $MR = HE =$ the ſum of the radii of the bodies; hence, they will be in contact when they arrive at thoſe points. Alſo, MR which joins their centers will paſs through the point of contact; and LC will be a tangent to them both.

Prop. LI.

226. *Having given the motions, the quantities of matter, and the radii of two ſpherical bodies which impinge obliquely upon each other, to find their motions after impact.*

Let LN be the plane which touches the bodies at the point of impact; produce AB, which joins the centers of the bodies, indefinitely both ways; through the centers A and B, draw EAF, GBH parallel to LN; let CA, DB repreſent the velocities of the bodies before impact; reſolve CA into the two CI, IA*, of which CI is parallel, and IA perpendicular to LN; alſo reſolve

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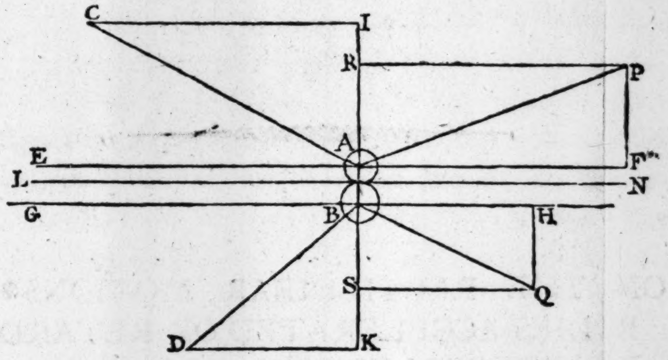
* Vid. Art. 43.

DB into two, *DK* parallel to *LN*, and *KB* perpendicular to it. Then *CA* and the angle *CAI*, which the direction of *A*'s motion makes with *AI* perpendicular to *LN*, being known, *CI* and *IA* are known; in the ſame manner, *DK* and *KB* are known. Now *CI*, *DK*, which are parallel to the plane *LN*, will not be affected by the impact; and *IA*, *KB*, which are perpendicular to it, are the velocities with which the bodies impinge directly upon each other, and their effects may be calculated by Prop. 44, when the bodies are perfectly hard; and by Prop. 47, when they are elaſtic. Let *AR* and *BS* be the velocities of the bodies after impact, thus determined; take *AF*=*CI*, and *BH*=*DK*; complete the parallelograms *RF*, *SH*, and draw the diagonals *AP*, *BQ*; then the bodies will deſcribe the lines *AP*, *BQ* after impact, and in the ſame time that they deſcribed *CA*, *DB* before impact.



$s \propto TV$

$V \propto FT$

$s \propto FT^2$

$s \propto \frac{V^2}{F}$

SECTION VII.

ON THE RECTILINEAR MOTIONS OF BODIES ACCELERATED OR RETARDED BY UNIFORM FORCES.

PROP. LII.

227. *If a body be impelled in a right line by an uniform force, the velocity communicated to it is proportional to the time of it's motion* *.

The accelerating force is meaſured by the velocity uniformly generated in a given time (Art. 21), and in this caſe, the force is invariable, by the ſuppoſition; therefore, equal increments of velocity are always generated in equal times (Art. 20); if then in the time t the velocity a be generated, in the time mt the velocity ma is generated; that is, the velocity generated is proportional to the time (Alg. Art. 193).

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* By *force*, in this and the following propoſitions, we underſtand the *accelerating force*, no regard being paid to the quantity of matter moved, unleſs it be expreſsly mentioned. Alſo, the direction in which the force acts, to generate or deſtroy velocity, is ſuppoſed to coincide with the direction of the motion.

PROP. LIII.

228. *If bodies be impelled in right lines by different uniform forces, the velocities generated in any times are proportional to the forces and times jointly.*

Let F and f be the forces, T and t the times of their action, V and v the velocities generated; alſo, let x be the velocity generated by the force f in the time T; then,

$$V : x :: F : f \text{ (Art. 21)};$$
$$x : v :: T : t \text{ (Art. 227)};$$

by compoſition, $V : v :: FT : ft$; that is, the velocities generated are proportional to the forces and times jointly (Alg. Art. 195).

Ex. If a force repreſented by unity, generate a velocity repreſented by $2m$, in one ſecond of time, what velocity will the force F generate in T ſeconds?

Since $V \propto FT$, we have $1 \times 1 : FT :: 2m : 2mFT$, the velocity required.

229. Cor. Since $V \propto FT$, $T \propto \frac{V}{F}$ (Alg. Art. 205).

PROP. LIV.

230. *If a body's motion be retarded by an uniform force, the velocity deſtroyed in any time is equal to that which would be generated in the ſame time, were the motion accelerated by the ſame force.*

The

The force impreſſed is the ſame, by the ſuppoſition, whether the body move in the direction of the force, or in the oppoſite direction; therefore, the velocity generated in the former caſe, is equal to the velocity deſtroyed, in the ſame time, in the latter (Art. 29).

231. Cor. 1. Hence, the velocity deſtroyed by an uniform force is proportional to the time of it's action.

For, the velocity generated by the action of the force is in that ratio (Art. 227).

232. Cor. 2. The velocities deſtroyed by different uniform forces, are proportional to the forces and times jointly (Art. 228).

Prop. LV.

233. *If a body be moved through any ſpace, from a ſtate of reſt, by the action of an uniform force, and then be projected in the oppoſite direction with the velocity acquired, and move till that velocity is deſtroyed, the whole ſpaces deſcribed in the two caſes are equal.*

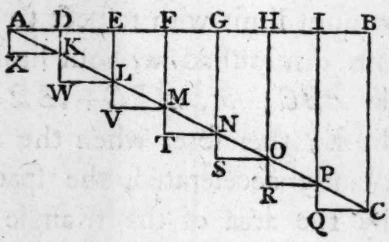
The velocity generated in any time, is equal to the velocity deſtroyed in the ſame time by the action of the ſame force (Art. 230); hence, the whole times of motion, in the two caſes, are equal; alſo, if equal times be taken, from the beginning of the motion in the former caſe, and from the end of the motion in the latter, the velocities at thoſe inſtants are equal. Since then the whole times of motion are equal, and alſo the velocities at all correſponding points of time, the whole ſpaces deſcribed are equal.

Prop.

Prop. LVI.

234. *If a body be moved from a ſtate of reſt by the action of an uniform force, the ſpace deſcribed, reckoning from the beginning of the motion, varies as the ſquare of the time, or as the ſquare of the laſt acquired velocity.*

Take *AB* to repreſent the time of the body's motion; draw *BC* at right angles to *AB*, and let *BC*



repreſent the laſt acquired velocity; join *AC*; divide the time *AB* into ſmall equal portions *AD*, *DE*, *EF*, *FG*, &c. and from the points *D*, *E*, *F*, *G*, &c. draw *DK*, *EL*, *FM*, *GN*, &c. parallel to *BC*, meeting *AC* in the points *K*, *L*, *M*, *N*, &c. complete the parallelograms *DX*, *EW*, *FV*, *GT*, &c.

Then, in the ſimilar triangles *ABC*, *ADK*, we have *AB* : *AD* :: *BC* : *DK*, and ſince *BC* repreſents the velocity acquired in the time *AB*, *DK* repreſents the velocity acquired in the time *AD* (Art. 227); in the ſame manner it appears, that *EL*, *FM*, *GN*, &c. repreſent the velocities generated in the times *AE*, *AF*, *AG*, &c. Now, if the body move with the uniform velocity *DK*, during the time *AD*, and with the uniform velocities *EL*, *FM*, *GN*, &c. during the times *DE*, *EF*, *FG*, &c. reſpectively, the ſpaces deſcribed may properly be repreſented by the rectangles DX, EW, FV, GT, &c. (Art. 16); therefore, the whole ſpace deſcribed, on this ſuppoſition, will be repreſented by the ſum of the rectangles, or by the triangle ABC together with the ſum of the triangles AXK, KWL, LVM, MTN, &c. that is, becauſe the baſes of theſe ſmall triangles are reſpectively equal to IB, and the ſum of their altitudes is equal to BC, by the triangle ABC together with half the rectangle BQ. Let the intervals AD, DE, EF, FG, &c. be diminiſhed without limit with reſpect to AB, and the rectangle BQ is diminiſhed without limit with reſpect to the triangle ABC; or, $ABC+\frac{1}{2}BQ$ approaches to ABC as it's limit; therefore, when the motion of the body is conſtantly accelerated, the ſpace deſcribed is repreſented by the area of the triangle ABC. The ſpace deſcribed in any other time AG, reckoning from the beginning of the motion, is repreſented, on the ſame ſcale, by the area of the triangle AGN; and becauſe theſe triangles are ſimilar, the ſpace deſcribed in the time AB : the ſpace deſcribed in the time $AG :: AB^2 : AG^2$.

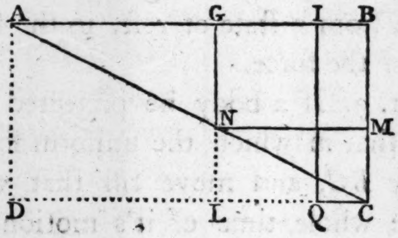
Alſo, BC, GN repreſent the velocities generated in the times AB, AG; and from the ſame ſimilar triangles, the ſpace deſcribed in the time AB : the ſpace deſcribed in the time $AG :: BC^2 : GN^2$.

235. Ex. If a body be accelerated from a ſtate of reſt by an uniform force, and deſcribe m feet in the firſt ſecond of time, it will deſcribe $4m$, $9m$, $16m$, ... mT^2 feet, in the 2, 3, 4, ... T firſt ſeconds.

236. Cor. 1. The ſpace deſcribed, reckoning from the beginning of the motion, is half that which would

be deſcribed in the ſame time with the laſt acquired velocity continued uniform.

Complete the parallelogram BD; then, it appears from the propoſition, that the ſpace deſcribed in the time AB, reckoning from the beginning of the motion : the ſpace deſcribed in the time IB with the uniform velocity BC :: the triangle ABC : BQ. Alſo, the ſpace deſcribed in the time IB, with the uniform velocity BC : the ſpace deſcribed in the time AB, with the ſame uniform velocity :: IB : AB (Art. 13) :: BQ : BD; and by compounding theſe two proportions, we have, the ſpace deſcribed in the time AB, when the body's motion is accelerated from a ſtate of reſt : the ſpace deſcribed in the ſame time with the laſt acquired velocity continued uniform :: the triangle ABC : the rectangle BD :: 1 : 2.



237. Cor. 2. The ſpace deſcribed in the time GB is repreſented by the area $GBCN$; or, if NM be drawn parallel to GB, by the rectangle GM together with the triangle NMC. Now, GM repreſents the ſpace which a body would deſcribe in the time GB, with the uniform velocity GN; and the triangle NMC, which is ſimilar to the triangle ABC, repreſents the ſpace through which the body would be moved from a ſtate of reſt, by the action of the force, in the time GB;

GB; thus, the ſpace deſcribed in any time, when a body is projected in the direction of the force, is equal to the ſpace which it would have deſcribed, in that time, with the firſt velocity continued uniform, together with the ſpace through which it would have been moved from a ſtate of reſt, in the ſame time, by the action of the force.

238. Cor. 3. If a body be projected in a direction oppoſite to that in which the uniform force acts, with the velocity *BC*, and move till that velocity is deſtroyed, the whole time of it's motion is repreſented by *BA* (Art. 230), and the ſpace deſcribed by the area *ABC* (Art. 233.)

Alſo, the ſpace deſcribed in the time *BG* is repreſented, on the ſame ſcale, by the area *BGNC*, that is, by the rectangle *BL* diminiſhed by the triangle *CLN*, or *CNM*. Thus it appears, that the ſpace deſcribed in the time *BG*, is equal to that which would have been deſcribed with the firſt velocity continued uniform during that time, diminiſhed by the ſpace through which the body would have been moved from a ſtate of reſt, in the ſame time, by the action of the uniform force.

Prop. LVII.

239. *When bodies are put in motion by uniform forces, the ſpaces deſcribed in any times, reckoning from the beginning of the motion in each caſe, are proportional to the times and laſt acquired velocities jointly.*

Let S and s be the ſpaces deſcribed in the times T and t; V and v the velocities acquired; then $2S$ and $2s$ are the ſpaces which would be deſcribed in the times T and t, with the uniform velocities V and v (Art. 236); and the ſpaces deſcribed with uniform velocities are proportional to the times and velocities jointly (Art. 14); hence,

$2S : 2s :: TV : tv$,

or $S : s :: TV : tv$ (Alg. Art. 184);

that is, $S \propto TV$ (Alg. Art. 195).

240. Cor. Hence, the times vary as the ſpaces directly and the laſt acquired velocities inverſely.

PROP. LVIII.

241. *The ſpaces deſcribed vary alſo as the forces and ſquares of the times; or as the ſquares of the velocities directly and the forces inverſely.*

In general, $S \propto TV$ (Art. 239); and $V \propto FT$ (Art. 228); hence, $TV \propto FT^2$ (Alg. Art. 203); therefore, $S \propto FT^2$. Alſo, $T \propto \frac{V}{F}$; therefore, $TV \propto \frac{V^2}{F}$, and conſequently, $S \propto \frac{V^2}{F}$.

242. Cor. 1. If T be given, $S \propto F$; that is, the ſpaces deſcribed in equal times, by bodies which are put in motion by uniform forces, are proportional to thoſe forces.

243. Cor. 2. Since $S \propto \frac{V^2}{F}$, we have $V^2 \propto FS$ (Alg. Art. 203); that is, the ſquares of the velocities communicated are proportional to the forces and ſpaces deſcribed jointly.

244. Cor. 3. If V be given, $S \propto \frac{1}{F}$.

245. Cor. 4.

245. Cor. 4. When bodies in motion are retarded by uniform forces, and move till their whole velocities are deſtroyed, the ſpaces deſcribed vary as the forces and ſquares of the times; or, as the ſquares of the firſt velocities directly and the forces inverſely.

For, the time in which any velocity is deſtroyed, is equal to the time in which it would be generated by the ſame force; alſo, the ſpaces deſcribed, on ſuppoſition that the body in the latter caſe is moved from a ſtate of reſt, are equal (Art. 233); therefore, the ſame expreſſions which repreſent the relations of the forces, ſpaces, times, and velocities, in accelerated motions, repreſent them alſo when the motions are retarded, and the bodies move till their whole velocities are deſtroyed.

Thus, when bodies are made to impinge upon banks of earth, ſand, &c. where the retarding forces are invariable, the depths to which they ſink, or the whole ſpaces deſcribed, are as the ſquares of the firſt velocities directly and the forces inverſely; and the reſiſting forces are as the ſquares of the firſt velocities directly and the ſpaces inverſely.

Prop. LIX.

246. *If a body be moved from a ſtate of reſt by the action of an uniform force, the ſpaces deſcribed in equal ſucceſſive portions of time, reckoned from the beginning of the motion, are as the odd numbers* 1, 3, 5, 7, 9, *&c.*

If m be the ſpace deſcribed in the firſt portion of time, $4m$ will be the ſpace deſcribed in the two firſt portions (Art. 235); therefore $4m - m$, or $3m$, will be the ſpace deſcribed in the ſecond portion alone. Alſo, $9m$ will be the ſpace deſcribed in the three firſt portions of time, and conſequently $9m - 4m$, or $5m$, will be the ſpace deſcribed in the third portion, &c. Thus the ſpaces deſcribed, in the equal ſucceſſive portions of time, are m, $3m$, $5m$, $7m$, $9m$, &c. which are as the odd numbers 1, 3, 5, 7, 9, &c.

247. Cor. When a body is retarded by an invariable force, the ſpaces deſcribed in equal portions of time, reckoning from the end of the motion, are as the odd numbers 1, 3, 5, 7, 9, &c.

For, when a body moves till it's whole motion is deſtroyed by an uniform force, the ſpace deſcribed in any time is equal to that which would be deſcribed in the correſponding time, were the body moved from a ſtate of reſt by the action of the ſame force (Vid. Art. 233).

Prop. LX.

248. *The force of gravity, at any given place, is an uniform force, which always acts in a direction perpendicular to the horizon, and accelerates all bodies equally.*

The ſame body will by it's gravity always produce the ſame effect under the ſame circumſtances; thus, it will, at the ſame place, bend the ſame ſpring in the ſame degree; it will alſo fall through the ſame ſpace in the ſame time, if the reſiſtance of the air be removed; therefore, the force of gravity is uniform. Alſo, all bodies which fall freely by this force, deſcend in lines perpendicular to the horizon; and, in an exhauſted receiver, they *all* fall through the ſame ſpace in the ſame time; conſequently, gravity acts in a direction perpendicular to the horizon (Art. 29), and accelerates *all* bodies equally (Art. 242).

It is found by experiments made on the deſcent of heavy bodies, and on the oſcillations of bodies in ſmall circular arcs (Art. 302), that every body which falls freely in vacuo by the force of gravity, deſcends from reſt through $16\frac{1}{12}$ feet in one ſecond.

This fact being eſtabliſhed, every thing relating to the deſcent of bodies when they are accelerated by the force of gravity, and to their aſcent when they are retarded by that force, ſuppoſing the motions to be in vacuo, may be deduced from the foregoing propoſitions.

1ſt. When a body falls by the force of gravity, the velocity acquired in any time, as T'', is ſuch as would carry it uniformly over $2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

Since a body falls $16\frac{1}{12}$ feet in $1''$, it acquires a velocity which would carry it uniformly through $32\frac{1}{6}$ feet in $1''$ (Art. 236); and when a body is accelerated by a given invariable force, the velocity generated is proportional to the time (Art. 227); therefore $1'' : T'' :: 32\frac{1}{6} : 32\frac{1}{6}\,T$, the velocity acquired in T''; that is, the velocity acquired is ſuch as would carry the body uniformly over $32\frac{1}{6}\,T$ feet in $1''$. Let V be the velocity acquired, and $m = 16\frac{1}{12}$, then $V = 2mT$.

2d. The ſpace fallen through in T'', reckoned from the beginning of the motion, is mT^2 feet.

For $S \propto T^2$ (Art. 234); therefore $1^2 : T^2 :: m : mT^2$, the ſpace deſcribed in T''. That is, $S = mT^2$.

Ex. 1. In $3''$ a body falls $9m$, or $9 \times 16\frac{1}{12} = 144\frac{3}{4}$ feet.

Ex.

$s = \frac{v^2}{4m}$ ∴ $v^2 = s \times 4m$.

$v = \sqrt{s \times 4m}$

$= \sqrt{r \times 4m}$

2

$= \sqrt{r \times 2}$

Ex. 2. In $\frac{1}{2}''$ a body falls $\frac{m}{4}$, or $16\frac{1}{12} \times \frac{1}{4} = 4\frac{1}{48}$ feet.

3d. The ſpace fallen through to acquire the velocity V, is $\frac{V^2}{4m}$ feet.

For, $S \propto V^2$ (Art. 234); therefore, $\overline{2m}|^2 : V^2 :: m : S$, and $S = \frac{V^2}{4m}$ feet.

Ex. If a body fall from reſt till it has acquired a velocity of 20 feet per ſecond, the ſpace fallen through is $\frac{20 \times 20}{64\frac{1}{3}}$, or 6.21 feet, nearly.

From the three preceding expreſſions, $V = 2mT$; $S = mT^2$; and $S = \frac{V^2}{4m}$; any one of the quantities S, T, V, being given, the other two may be found.

Ex. To find the time in which a body will fall 90 feet; and the velocity acquired.

Since $S = mT^2$, $T^2 = \frac{S}{m}$, and $T = \sqrt{\frac{S}{m}}$; in this caſe, $T = \sqrt{\frac{90}{16\frac{1}{12}}} = 2.36$ ſeconds, nearly.

Alſo, $S = \frac{V^2}{4m}$; therefore, $V = 2\sqrt{mS}$; in this caſe, $V = 2\sqrt{16\frac{1}{12} \times 90} = 76$ feet per ſecond, nearly.

4th. If a body fall from reſt by the force of gravity, the ſpaces deſcribed in any equal ſucceſſive portions of time, reckoning from the beginning of the motion, are as the numbers 1, 3, 5, 7, &c. Thus, the ſpaces fallen through in the 1st, 2nd, 3rd, 4th ſeconds are $16\frac{1}{12}$, $3 \times 16\frac{1}{12}$, $5 \times 16\frac{1}{12}$, $7 \times 16\frac{1}{12}$ feet, reſpectively. Alſo, if a body, projected upwards, move till it's whole velocity is deſtroyed, the ſpaces deſcribed in equal ſucceſſive portions of time are as the numbers 1, 3, 5, 7, &c. taken in an inverted order. Thus, if the velocity be wholly deſtroyed in $4''$, the ſpaces deſcribed in the 1st, 2nd, 3rd, 4th ſeconds are $7 \times 16\frac{1}{12}$, $5 \times 16\frac{1}{12}$, $3 \times 16\frac{1}{12}$, $16\frac{1}{12}$ feet, reſpectively.

5th. If a body begin to move in the direction of gravity with any velocity, the whole ſpace deſcribed in any time is equal to the ſpace through which the firſt velocity would carry the body, together with the ſpace through which it would fall by the force of gravity in that time (Art. 237).

Ex. If a body be projected perpendicularly downwards, with a velocity of 20 feet per ſecond, to find the ſpace deſcribed in $4''$.

The ſpace deſcribed in $4''$ with the firſt velocity is 4×20, or 80 feet; and the ſpace fallen through in $4''$ by the action of gravity is $16\frac{1}{12} \times 16$, or $257\frac{1}{3}$ feet; therefore, the whole ſpace deſcribed is $80 + 257\frac{1}{3}$, or $337\frac{1}{3}$ feet.

6th. If a body be projected perpendicularly upwards, the height to which it will aſcend in any time is equal to the ſpace through which it would move with the firſt velocity continued uniform, diminiſhed by the ſpace through which it would fall by the action of gravity in that time (Art. 238).

Ex. 1. To what height will a body riſe in 3″, if projected perpendicularly upwards with a velocity of 100 feet per ſecond?

The ſpace which the body would deſcribe in 3″, with the firſt velocity, is 300 feet; and the ſpace through which the body would fall by the force of gravity in 3″, is $16\frac{1}{12} \times 9$, or $144\frac{3}{4}$ feet; therefore the height required is $300 - 144\frac{3}{4}$, or $155\frac{1}{4}$ feet.

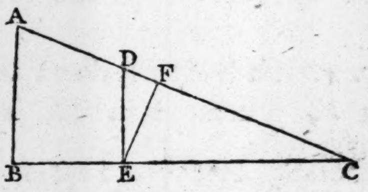
Ex. 2. If a body be projected perpendicularly upwards with a velocity of 80 feet per ſecond, to find it's place at the end of 6″.

The ſpace which would be deſcribed in 6″, with the firſt velocity, is 480 feet, and the ſpace fallen through in the ſame time is $16\frac{1}{12} \times 36$, or 579 feet; therefore the diſtance of the body from the point of projection, at the end of 6″, is $480 - 579$, or -99 feet. The negative ſign ſhews that the body will be below the point of projection (Vid. Alg. Art. 474).

Prop. LXI.

249. *The force which accelerates or retards a body's motion upon an inclined plane, is to the force of gravity, as the height of the plane to it's length.*

Let *AC* be the plane, *BC* it's baſe, parallel to the



horizon, *AB* it's perpendicular height, *D* the place of a body upon it. From the point D draw DE parallel to AB, and take DE to repreſent the force of gravity; from E draw EF perpendicular to AC. Then the whole force DE is equivalent to the two DF, FE, of which FE is perpendicular to the plane, and conſequently, is ſupported by the plane's reaction (Art. 116); the other force DF, not being affected by the plane, is wholly employed in accelerating or retarding the motion of the body in the direction of the plane; therefore, the accelerating force : the force of gravity :: $DF : DE$ :: (from the ſimilar triangles DEF, ABC) $AB : AC$.

250. Cor. 1. Since the accelerating force, on the ſame plane, is in a given ratio to the force of gravity, it is an uniform force.

251. Cor. 2. If H be the height of an inclined plane, L it's length, and the force of gravity be repreſented by unity, the accelerating force on the inclined plane is repreſented by $\frac{H}{L}$.

For, the accelerating force : the force of gravity (1) :: $H : L$; therefore the accelerating force $= \frac{H}{L}$.

252. Cor. 3. Since $H : L$ :: the ſine of the plane's inclination : the radius, $\frac{H}{L}$, or the accelerating force, varies as the ſine of the plane's inclination to the horizon.

253. Cor. 4. If a body fall down an inclined plane, the velocity V, generated in T'', is ſuch as would carry it uniformly over $\frac{H}{L} \times 2mT$ feet in $1''$; where $m = 16\frac{1}{12}$.

In general, $V \propto FT$ (Art. 228); therefore, the velocity acquired when a body falls by the force of gravity : the velocity acquired on the inclined plane :: the product of the numbers which repreſent the force and time in the former caſe : the product of the numbers which repreſent them in the latter*; alſo, the force of gravity being repreſented by unity, the accelerating force upon the plane is $\frac{H}{L}$, and the velocity generated by the force of gravity in 1″ is $2m$; therefore, $2m : V :: 1 \times 1 : \frac{H}{L} \times T$; and $V = \frac{H}{L} \times 2mT$†.

Ex. Thus, if the length of an inclined plane be twice as great as it's height, a body which falls down this plane will, in 3″ acquire a velocity of $\frac{1}{2} \times 32\frac{1}{6} \times 3$, or $48\frac{1}{4}$ feet per ſecond.

254. Cor. 5. The ſpace fallen through in T'', from a ſtate of reſt, is $\frac{H}{L} \times mT^2$ feet.

In general, $S \propto FT^2$ (Art. 241); therefore, the ſpace through which a body falls by the action of gravity in 1″ : the ſpace through which it falls down the inclined plane in T'' :: the product of the numbers which repreſent the force and ſquare of the time in the former caſe : the product of the numbers which repreſent them in the latter; or, if S be the ſpace deſcribed upon the plane, $m : S :: 1 \times 1^2 : \frac{H}{L} \times T^2$, and $S = \frac{H}{L} \times mT^2$.

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* Vid. note, page 12.

† In this, and the following articles, the planes are ſuppoſed to be perfectly ſmooth, and the reſiſtance of the air inconſiderable.

Ex. 1. If $L = 2H$, the ſpace through which a body falls in 3″ is $\frac{1}{2} \times 16\frac{1}{12} \times 9$, or $72\frac{3}{8}$ feet.

Ex. 2. To find the time in which a body will deſcend 12 feet down this plane.

Since $S = \frac{H}{L} \times mT^2$, $T^2 = \frac{L \times S}{H \times m} =$ (in this caſe) $\frac{2}{1} \times 12 \times \frac{1}{16\frac{1}{12}} = 1.49$; and $T = 1.2$, nearly.

255. Cor. 6. The ſpace through which a body muſt fall, from a ſtate of reſt, to acquire the velocity V, is $\frac{L}{H} \times \frac{V^2}{4m}$ feet.

In general, $S \propto \frac{V^2}{F}$ (Art. 241); therefore, the ſpace through which the body falls by the force of gravity : the ſpace through which it falls down the plane :: $\frac{V^2}{F}$ in the former caſe : $\frac{V^2}{F}$ in the latter; and if m $(16\frac{1}{12})$ be the ſpace fallen through by the action of gravity, $2m$ is the velocity acquired; hence, $m : S :: \frac{4m^2}{1} : \frac{L}{H} \times V^2$; and $S = \frac{L}{H} \times \frac{V^2}{4m}$.

Ex. 1. If $L=2H$, and a body fall from a ſtate of reſt till it has acquired a velocity of 30 feet per ſecond, the ſpace deſcribed is $\frac{2}{1}\times\frac{900}{64\frac{1}{3}}=27.97$ feet, nearly.

Ex. 2. If a body fall 12 feet from a ſtate of reſt down this plane, to find the velocity acquired.

Since $S=\frac{L}{H}\times\frac{V^2}{4m}$, we have $V^2=4mS\times\frac{H}{L}=$ (in this caſe) $64\frac{1}{3}\times 12\times\frac{1}{2}=386$; hence, $V=19.6$ feet per ſecond, nearly.

Prop. LXII.

256. *The velocity which a body acquires in falling down the whole length of an inclined plane, varies as the ſquare root of the perpendicular height of the plane*.*

In general, when the force is uniform, $V^2\propto FS$ (Art. 243); in this caſe, $F\propto\frac{H}{L}$, and $S=L$, by the ſuppoſition; therefore $V^2\propto\frac{H}{L}\times L\propto H$; and $V\propto\sqrt{H}$ (Alg. Art. 202).

257. Cor. 1. When the heights of two inclined planes are equal, the velocities acquired in falling down their whole lengths are equal.

258. Cor. 2. The velocity which a body acquires in falling down the length of an inclined plane is equal to the velocity which it would acquire in falling down it's perpendicular height.

Prop.

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* Bodies, in this, and the ſubſequent propoſitions, are ſuppoſed to fall from a ſtate of reſt.

Prop. LXIII.

259. *The time of a body's deſcent down the whole length of an inclined plane, varies as the length directly, and as the ſquare root of the perpendicular height inverſely.*

In general, $S \propto TV$ (Art. 239); therefore, $T \propto \frac{S}{V}$; and in this caſe, $V \propto \sqrt{H}$ (Art. 256); conſequently, $T \propto \frac{S}{\sqrt{H}} \propto \frac{L}{\sqrt{H}}$.

260. Cor. 1. If the height, or the laſt acquired velocity be given, $T \propto L$.

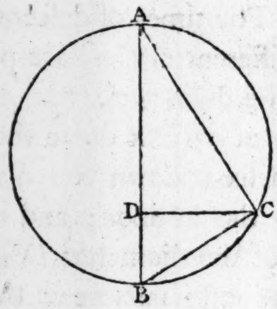
261. Cor. 2. If the inclination be given, or $H \propto L$, then $T^2 \propto \frac{L^2}{L} \propto L$, and $T \propto \sqrt{L}$. That is, the times of deſcent, down planes equally inclined to the horizon, vary as the ſquare roots of their lengths.

262. Cor. 3. The time of deſcent down an inclined plane, is to the time of falling down it's perpendicular height, as the length of the plane to it's height.

Prop. LXIV.

263. *If chords be drawn in a circle from the extremity of that diameter which is perpendicular to the horizon, the velocities which bodies acquire by falling down them are proportional to their lengths; and the times of deſcent are equal.*

Let ACB be the circle, AB it's diameter perpendicular to the horizon; BC a chord drawn from the extremity B of the diameter; join AC, and draw CD



perpendicular to AB, or parallel to the horizon. Then CB may be conſidered as an inclined plane whoſe perpendicular height is DB, and the velocity acquired in falling down it varies as $\sqrt{DB}$ (Art. 256). Now, from the ſimilar triangles DBC, ABC, $DB : CB :: CB : AB$; therefore $DB = \frac{CB^2}{AB}$, and $\sqrt{DB} = \frac{CB}{\sqrt{AB}}$; conſequently, $V \propto \frac{CB}{\sqrt{AB}}$, and AB is invariable; therefore $V \propto CB$.

Again, $T \propto \frac{S}{V}$ (Art. 240), and in this caſe, CB, which is the ſpace deſcribed, has been proved to be proportional to the velocity acquired; therefore $T \propto \frac{CB}{CB}$, or the time of deſcent is invariable.

264. Cor. 1. The time of deſcent down any chord CB, is equal to the time of deſcent down the diameter AB.

265. Cor. 2. In the ſame manner, the time of deſcent down AC is equal to the time of deſcent down AB;

AB; therefore the time of deſcent down *AC* is equal to the time of deſcent down *CB*.

266. Cor. 3. The times of deſcent down the chords thus drawn, in different circles, are proportional to the ſquare roots of the diameters.

For, the times of deſcent down the chords are equal to the times of deſcent down the diameters which are perpendicular to the horizon; and theſe times vary as the ſquare roots of the diameters (Vid. Art. 234).

267. When a body falls freely by the force of gravity, every particle in it is equally accelerated; that is, every particle deſcends towards the horizon with the ſame velocity; in this deſcent therefore, no rotation will be given to the body. The ſame may be ſaid when a body deſcends along a perfectly ſmooth inclined plane, if that part of the force which acts in a direction perpendicular to the plane (Art. 249), be ſupported; that is, if a perpendicular to the plane, drawn from the center of gravity of the body, cut the plane in a point which is in contact with the body. If this part of the force be not ſuſtained by the plane, the body will partly roll and partly ſlide, till this force is ſuſtained; and afterwards the body will wholly ſlide. When the lateral motion is entirely prevented by the adheſion of the body to the plane, we have before ſeen on what ſuppoſition the body will roll (Art. 186); if the adheſion be not ſufficient to prevent all lateral motion, this body will partly ſlide and partly roll; and to eſtimate the ſpace deſcribed, the time of it's motion, or the velocity acquired, we muſt have recourſe to other principles than thoſe above laid down. On this ſubject the Reader may conſult Mr. VINCE's *Plan of a Courſe of Lectures*, p. 39.

268. When a body falls freely by the force of gravity, or deſcends along a perfectly ſmooth inclined plane, the accelerating force is the ſame, whatever be the weight of the body (Arts. 248. 249); conſequently, the moving force, on either ſuppoſition, is proportional to the quantity of matter moved. In all caſes, the accelerating force varies as the moving force directly and the quantity of matter inverſely (Art. 24); and when the moving force and quantity of matter moved are invariable, the accelerating force is uniform, and it's effects may be eſtimated by the rules laid down in the firſt part of this Section.

Ex. If two bodies, whoſe weights are P and Q, be connected by a ſtring, and hung over a fixed pulley, to find how far the heavier P will deſcend in T''.

The moving force of gravity is proportional to the weight; if therefore P be taken to repreſent the moving force of the former body when it deſcends freely, Q will repreſent the moving force of the latter, and $P-Q$ will repreſent the moving force when the bodies are connected and oppoſe each other's motion; hence, neglecting the inertia of the ſtring and pulley, the accelerating force of gravity : the accelerating force in this caſe :: $\frac{P}{P} : \frac{P-Q}{P+Q} :: 1 : \frac{P-Q}{P+Q}$; and, ſince $FT^2 \propto S$, $1 \times 1^2 : \frac{P-Q}{P+Q} \times T^2 :: 16\frac{1}{12} : 16\frac{1}{12} \times \frac{P-Q}{P+Q} \times T^2$, the ſpace required.

SECTION

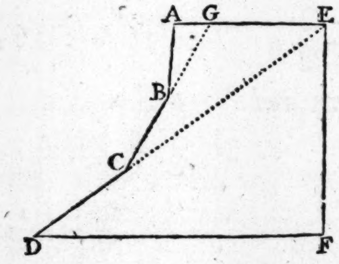
SECTION VIII.

ON THE OSCILLATIONS OF BODIES IN CYCLOIDS AND IN SMALL CIRCULAR ARCS.

Prop. LXV.

269. *If a body deſcend down a ſyſtem of inclined planes, the velocity acquired, on the ſuppoſition that no motion is loſt in paſſing from one plane to another, is equal to that which would be acquired in falling through the perpendicular height of the ſyſtem.*

Let $ABCD$ be the ſyſtem of planes; draw AE,



DF parallel to the horizon; produce CB, DC till they meet AE in G and E; and draw EF perpendicular to DF. Then the velocity acquired by a body in falling from A to B, is equal to that which it would acquire in falling from G to B, becauſe the planes AB, GB have the ſame perpendicular height (Art. 257); and ſince, by the ſuppoſition, no velocity is loſt in paſſing from one plane to another, the body will begin to deſcend down BC with the ſame velocity, whether it fall down AB or GB; conſequently, the velocity acquired at C will be the ſame on either ſuppoſition. Alſo, the velocity acquired at C is equal to that which would be acquired in falling down EC (Art. 257); and no velocity being loſt at C, the body will begin to deſcend down CD with the ſame velocity, whether it fall from A through B and C to D, or from E to D; and the velocity acquired in falling down ED is equal to the velocity acquired in falling through the perpendicular height EF (Art. 258); therefore, the velocity acquired in falling down the whole ſyſtem, is equal to the velocity acquired in falling through the perpendicular height of the ſyſtem.

PROP. [illegible]XVI.

270. *If a body fall from a ſtate of reſt down a curve ſurface which is perfectly ſmooth, the velocity acquired is equal to that which would [illegible] in falling from reſt through the ſame perpendicular height.*

When a body paſſes from one plane AB to another BC, the whole velocity : the quantity by which the velocity is diminiſhed :: radius : the verſed ſine of the $\angle ABG$ (Art 221); when, therefore, the angle ABG

is diminiſhed without limit, the velocity loſt is diminiſhed without limit; and if the lengths of the planes as well as their angles of inclination *ABG*, *BCE* be continually diminiſhed, the ſyſtem approximates to a curve as it's limit, in which no velocity is loſt; conſequently, the whole velocity acquired is equal to that which a body would acquire in falling through the ſame perpendicular altitude (Art. 269)*.

271. Cor. 1. If a body be projected up a curve, the perpendicular height to which it will riſe is equal to that through which it muſt fall to acquire the velocity of projection.

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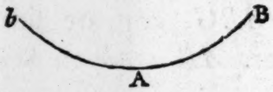
* When the chord of an arc is diminiſhed without limit with reſpect to the diameter, the verſed ſine is diminiſhed without limit with reſpect to the chord; becauſe, the diameter : the chord :: the chord : the verſed ſine; hence, the ratio of the diameter to the verſed ſine, and conſequently, the ratio of the radius to the verſed ſine, is, in this caſe, indefinitely greater than the ratio of the diameter to the chord. Let BC be one of the evaneſcent planes, V the velocity of the deſcending body at B, $V+v$ it's velocity at C; produce CB to G, and let GB be the ſpace through which the body muſt deſcend to acquire the velocity V; then, $V : V+v :: \sqrt{GB} : \sqrt{GB+BC}$; and when $GB : BC$ :: the radius : an evaneſcent chord, $V : V+v :: GB : GB+\frac{BC}{2}$ (Vide note, page 126); therefore, $V : v :: GB : \frac{BC}{2} :: 2GB : BC$. Alſo, V : the velocity loſt at B :: radius : the verſed ſine of the angle ABG. Hence it follows, that the ratio of V to the velocity loſt at B, is indefinitely greater than the ratio of V to the velocity acquired in the deſcent from B to C; and conſequently, the velocity loſt at B is indefinitely leſs than the velocity acquired in the deſcent from B to C; in the ſame manner, the velocity loſt at any other plane, is indefinitely leſs than the velocity acquired in the deſcent down that plane; therefore, the velocity loſt in the whole deſcent is indefinitely leſs than the whole velocity acquired.

For the body in it's aſcent will be retarded by the ſame degrees that it was accelerated in it's deſcent.

272. Cor. 2. If *BAb* be a curve in which the loweſt point is *A*, and the parts *AB*, *Ab* are ſimilar and equal, a body in falling down *BA* will acquire a velocity which will carry it to *b*; and ſince the velocities at all equal altitudes in the aſcent and deſcent are equal, the whole time of the aſcent will be equal to the time of deſcent.



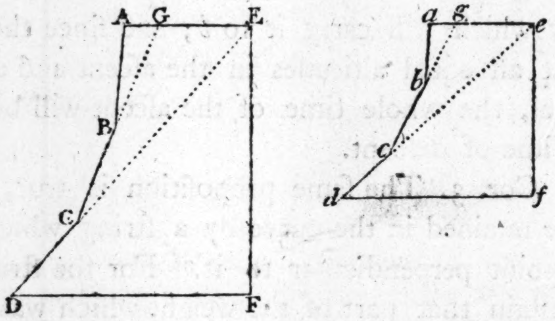
273. Cor. 3. The ſame propoſition is true, if the body be retained in the curve by a ſtring which is in every point perpendicular to it. For the ſtring will now ſuſtain that part of the weight which was before ſuſtained by the curve (Art. 117).

Prop. LXVII.

274. *The times of deſcent down ſimilar ſyſtems of inclined planes, ſimilarly ſituated, are as the ſquare roots of their lengths, on the ſuppoſition that no velocity is loſt in paſſing from one plane to another.*

Let *ABCD*, *abcd*, be two ſimilar ſyſtems of inclined planes, ſimilarly ſituated; that is, let *AB* : *ab* :: *BC* : *bc* :: *CD* : *cd*; the angles *ABC*, *BCD*, reſpectively equal to the angles *abc*, *bcd*; and the planes *AB*, *ab*, equally inclined to the horizon. Complete the figures as in the laſt propoſition; then, ſince *AB* : *ab* :: *BC*

$BC : bc :: CD : cd$, we have, $AB : ab :: AB+BC+CD : ab+bc+cd$ (Alg. Art. 183); and, $\sqrt{AB} : \sqrt{ab} :: \sqrt{AB+BC+CD} : \sqrt{ab+bc+cd}$. Alſo, ſince the angles ABC, abc are equal, their ſupplements, the angles ABG, abg are equal; and the angles of inclination to the horizon BAG, bag are equal; therefore, the triangles ABG, abg are ſimilar, and $AB : BG :: ab : bg$; alter. $AB : ab :: BG : bg :: BC : bc$; conſequently, $BG : bg :: BG+BC\ (GC) : bg+bc\ (gc) :: AB : ab$. In the ſame manner, $ED : ed :: AB : ab$. Then, becauſe the planes AB, ab, are equally inclined to the horizon, the time of deſcent down AB : the time down $ab :: \sqrt{AB} : \sqrt{ab}$ (Art. 261); and if bodies fall down GC, gc, the time down GC : the time down $gc :: \sqrt{GC} : \sqrt{gc} :: \sqrt{AB} : \sqrt{ab}$; alſo, the time down GB : the time down $gb :: \sqrt{GB} : \sqrt{gb} :: \sqrt{AB} : \sqrt{ab}$; hence, the whole time down GC : the whole time down gc :: the time down GB : the time down gb; therefore, the remainder, the time down BC : the remainder, the time down bc, in the ſame ratio, or as $\sqrt{AB} : \sqrt{ab}$ (Euc. 19. 5); and ſince, by the ſuppoſition, no motion

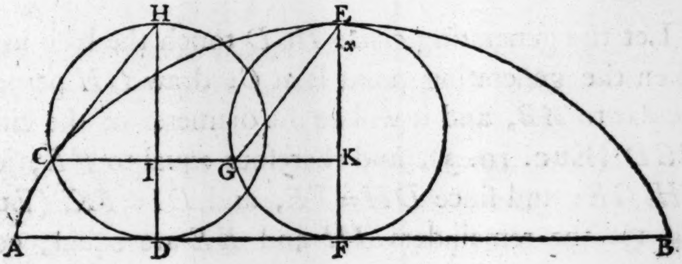


tion is loſt in paſſing from one plane to another, the times of deſcent down BC and bc are the ſame, whether the bodies deſcend from A and a, or from G and g; conſequently, when the bodies deſcend down the ſyſtems, the time down BC : the time down bc :: $\sqrt{AB} : \sqrt{ab}$. In the ſame manner it may be ſhewn that the time down CD : the time down cd :: $\sqrt{AB}$: : $\sqrt{ab}$. Hence, the time down AB : the time down ab :: the time down BC : the time down bc :: the time down CD : the time down cd; therefore, the time down $AB+BC+CD$: the time down $ab+bc+cd$ :: the time down AB : the time down ab :: $\sqrt{AB}$: $\sqrt{ab}$ (Algebra, Art. 183) :: $\sqrt{AB+BC+CD}$: $\sqrt{ab+bc+cd}$.

275. Cor 1. If the lengths of the planes, and their angles of inclination ABG, BCE, &c. be continually diminiſhed, the limits, to which theſe ſyſtems approximate, are ſimilar curves, ſimilarly ſituated, in which no velocity is loſt (Art. 270); hence, the whole times of deſcent down theſe curves will be as the ſquare roots of their lengths.

276. Cor. 2. The times of deſcent down ſimilar circular arcs, ſimilarly ſituated, are as the ſquare roots of the arcs, or as the ſquare roots of their radii.

277. Def. If a circle be made to *roll*, in a given



plane upon a ſtraight line *AB*, the point *C* in the circumference, which was in contact with *AB* at the beginning of the motion, will, in a revolution of the circle, deſcribe a curve *ACEB* called a *cycloid*.

The line *AB* is called the *baſe* of the cycloid.

The circle *HCD* is called the *generating circle*.

The line *FE*, which is drawn biſecting *AB* at right angles, and produced till it meets the curve in *E*, is called the *axis*, and the point *E*, the *vertex*, of the cycloid.

278. Cor. 1. The baſe *AB* is equal to the circumference of the generating circle; and *AF* to half the circumference.

279. Cor. 2. The axis *FE* is equal to the diameter of the generating circle.

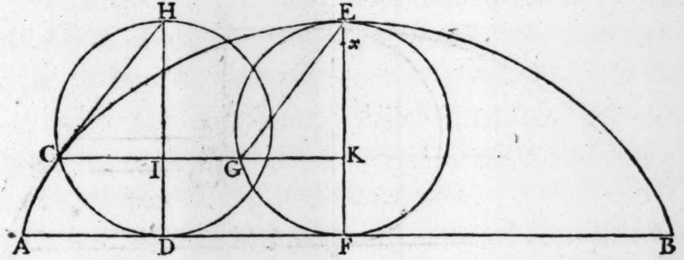
When the generating circle comes to *F*, draw the diameter *Fx*, which will be perpendicular to *AB* (Euc. 18. 3); and becauſe the circle has completed half a revolution, *x* is the generating point; that is, *x* is a point in the cycloid, or *x* coincides with *E*.

Prop. LXVIII.

280. *If a line* CGK, *drawn from a point* C *in the cycloid, parallel to the baſe* AB, *meet the generating circle, deſcribed upon the axis, in* G, *the circular arc* EG *is equal to the right line* CG.

Let the generating circle *HCD* touch the baſe in *D* when the generating point is at *C*; draw *DH* perpendicular to *AB*, and it will be the diameter of the circle *HCD* (Euc. 19. 3), and therefore equal to *FE*; join *CH*, *GE*; and ſince $DH = FE$, and $DI = FK$, (Euc. 34. 1), the remainders *IH* and *KE* are equal; conſequently,

ſequently, CI, which is a mean proportional between



HI and ID (Euc. Cor. 8. 6), is equal to KG, which is a mean proportional between EK and KF; to each of theſe equals add IG, and $CG = IK$. Alſo, CH, which is a mean proportional between IH and HD, is equal to GE, which is a mean proportional between EK and EF; therefore, the arc $CH =$ the arc GE (Euc. 28. 3); and ſince every point in CD has been ſucceſſively in contact with AD, $CD = AD$, and $HCD = AF$ (Art. 278); hence, the arc $CH = DF$; therefore, the arc $EG = DF = IK = CG$.

Cor. If the line KGC be always drawn perpendicular to EF the diameter of the circle EGF, and GC be taken equal to the arc EG, the locus of the point C is a cycloid, whoſe axis is EF.

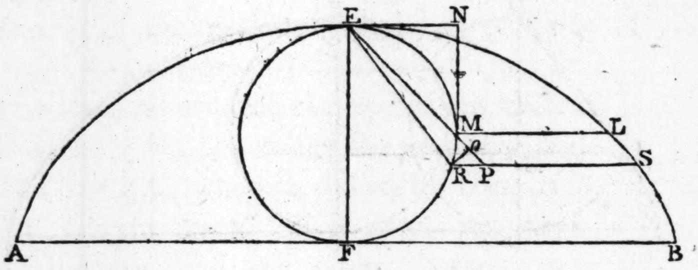
Prop. LXIX.

281. *If a line* LM, *drawn from* L *parallel to the baſe* AB, *meet the generating circle deſcribed upon the axis in* M, *and* EM *be joined, the tangent to the cycloid at the point* L *is parallel to the chord* EM.

Draw SR parallel and indefinitely near to LM; join EM, RM, SL; produce EM till it meets SR in P; draw EN, MN touching the circle in E and M, and meeting each other in N.

Then

Then ſince RM is ultimately in the direction of the



tangent MN (Newt. Lem. 6), the angles RMP, EMN are equal; and becauſe EN is parallel to RS (Euc. 18. 3), the angles MPR, MEN are equal; therefore, the triangles EMN, RMP are equi-angular, and $EN : MN :: RP : RM$; and ſince $EN = NM$, $RP = RM =$ the arc RM (Newt. Lem. 7). Again, ſince the arc $EMR = RS$ (Art. 280), and $RM = RP$, the remainders, the arc EM and the right line PS are equal; alſo, $ML =$ the arc EM; therefore, $PS = ML$; conſequently, SL is equal, and parallel to PM (Euc. 33. 1*); and ſince SL is ultimately in the direction of the tangent at L (Newt. Lem. 6), MP, or EM, is parallel to the tangent at L.

282. Cor. The tangent to the cycloid at B or A, is perpendicular to AB.

Prop. LXX.

283. *The ſame conſtruction being made, the cycloidal arc* EL *is double of* EM *the correſponding chord of the generating circle deſcribed upon the axis.*

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* The propoſition may be juſtly applied, becauſe the difference between LM and SP is evaneſcent with reſpect to MP, or LS.

Join ER, and in EP take $Eo=ER$; join Ro. Then, when the arc MR, and conſequently the angle MER, is diminiſhed without limit, the ſum of the angles ERo, EoR approximates to two right angles as it's limit; and theſe angles are equal to each other; therefore, each of them is a right angle; and ſince the angles RMo, RoM are reſpectively equal to RPo, RoP, and Ro is common to the triangles RoM, RPo, $Mo=oP$, and $MP=2Mo$; alſo, Mo $(ER-EM)$ is the quantity by which the chord EM increaſes, whilſt the cycloidal arc EL increaſes by LS; and it appears from the demonſtration of the laſt propoſition that $MP=LS=$ arc LS (Newt. Lem. 7); therefore, the arc $LS=2Mo$; or, the cycloidal arc EL increaſes twice as faſt as the correſponding chord EM; and they begin together at E; conſequently, the cycloidal arc EL is double of EM, the correſponding chord of the generating circle.

284. Cor. The whole ſemi-cycloidal arc EB is equal to twice the axis EF.

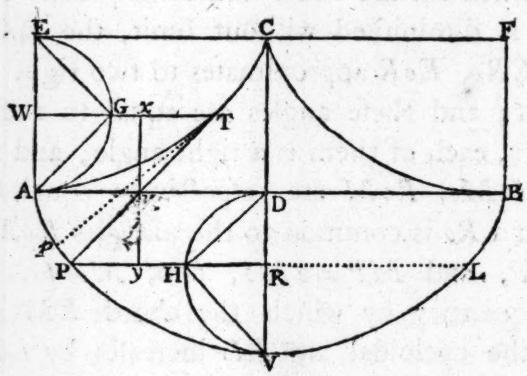
Prop. LXXI.

285. *To make a body oſcillate in a given cycloid.*

Let AVB be the given cycloid, placed with it's vertex downwards, and it's axis DV perpendicular to the horizon. Produce VD to C, making $DC=VD$; complete the rectangles DE, DF; upon AE deſcribe a ſemicircle AGE, and with A as the generating point, and baſe EC, deſcribe a ſemi-cycloid ATC; this will paſs through the point C, becauſe the ſemi-circumference $AGE=DHV=AD=EC$; in the ſame manner,

deſcribe

deſcribe an equal ſemi-cycloid between C and B.



Then, if a body P be ſuſpended from C by a ſtring whoſe length is CV or CTA (Art. 284), and made to vibrate between the cycloidal cheeks CA, CB, it will always be found in the cyloid AVB.

Let the ſtring be brought into the ſituation CTP, and ſince it is conſtantly ſtretched by the gravity, and the centrifugal force of P, it will be a tangent to the cycloid at the point T where it leaves the curve. From T and P draw TGW, PHR parallel to AD; join AG, GE, DH, HV; and through K draw xKy perpendicular to TG or PH. Then, ſince the chord AG is parallel to TP (Art. 281), and TG is parallel to AK, the figure GK is a parallelogram, and $AG=TK$, $GT=AK$; and becauſe the length of the ſtring is equal to CTA, and the part CT is common to the ſtring and the cycloidal arc, $TP=AT=2AG$ (Art. 283) $=2TK$; or $TK=KP$; hence, the triangles TKx, PKy are ſimilar and equal, and $Kx=Ky$; alſo, $Kx=AW$ and $Ky=DR$; therefore $AW=DR$; and $AE=DV$; hence, the arc $AG=$ the arc DH; and the angle $GEA=$

the angle *HVD*; or, the angle *GAK*= the angle *KDH* (Euc. 32. 3); conſequently, *AG* is parallel to *DH*; and therefore, *TP* is parallel to *DH*, and the figure *KPHD* is a parallelogram; hence, *KD*=*PH*. Again, ſince the arc *AG*=*GT* (Art. 280)=*AK*, the arc *DH* =*AK*; and the ſemi-circumference *DHV*=*AD*; therefore, the arc *VH*=*KD*=*HP*; that is, *P* is in the cycloid, whoſe axis is *DV*, and vertex *V* (Art. 280).

286. Cor. 1. Since *DH* is parallel to *TP*, and *VH* to the tangent at *P*, the angle contained between *TP* and the tangent, or between *TP* and the curve, is equal to the angle *DHV*; that is, *TP* is always perpendicular to the curve.

287. Cor. 2. If *Pp* be an evaneſcent arc, the perpendiculars to the curve at *P* and *p*, ultimately meet in *T*; and *Pp* may be conſidered as a circular arc whoſe radius is *TP*.

288. Cor. 3. An evaneſcent arc at the vertex of the cycloid may be conſidered as a circular arc whoſe radius is *CV*.

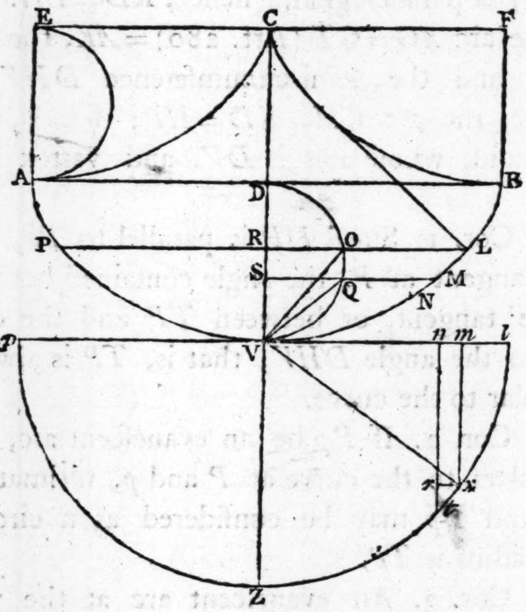
289. Def. If a body begin to deſcend in a curve, from any point, and again aſcend till it's velocity is deſtroyed (Art. 272), the time in which the motion is performed is called *the time of an oſcillation.*

Prop. LXXII.

290. *If a body, vibrating in the cycloid* AVB, *begin to deſcend from* L, *the velocity acquired at any point* M *varies as* $\sqrt{VL^2 - VM^2}$; *or, as the right ſine of a circular arc whoſe radius is equal to* VL, *and verſed ſine to* LM.

From

From the points L and M, draw LOR, MQS, at right angles to DV, meeting the circle DOV in O



and Q; join OV, QV; with the radius $Vl=VL$, deſcribe the ſemicircle lZp, and take $lm=LM$; draw mx, VZ, at right angles to Vl; and join Vx.

The velocity acquired in the deſcent from L to M, is equal to the velocity acquired in falling from R to S (Art. 270); and therefore it varies as $\sqrt{RS}$ (Art. 241); that is, $\propto \sqrt{RV-SV} \propto \sqrt{DV\times RV - DV\times SV}$ (becauſe DV is invariable), $\propto \sqrt{VO^2 - VQ^2} \propto \sqrt{4VO^2 - 4VQ^2} \propto \sqrt{VL^2 - VM^2}$ (Art. 283), $\propto \sqrt{Vl^2 - Vm^2} \propto \sqrt{Vx^2 - Vm^2} \propto \sqrt{mx^2} \propto mx$.

291. Cor. The velocity at M : the velocity at V :: $mx : VZ :: mx : Vx$.

Prop.

Prop. LXXIII.

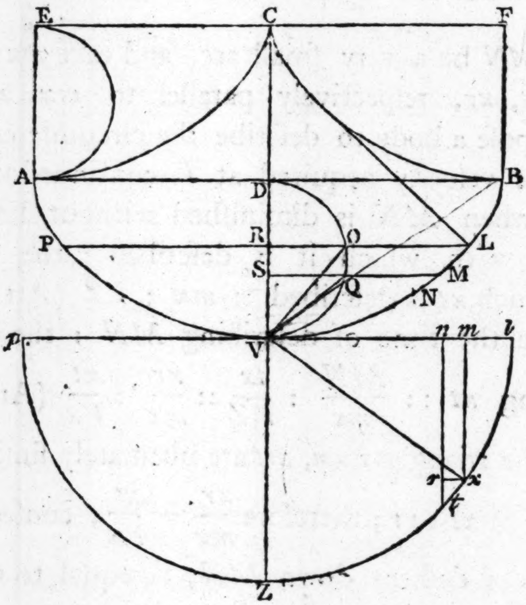
292. *The time of an oſcillation in the arc* LVP *is equal to the time in which a body would deſcribe the ſemi-circumference* lZp *with the velocity acquired at* V *continued uniform.*

Let MN be a very ſmall arc, and take $mn = MN$; draw nt, xr, reſpectively parallel to mx and Vl; and ſuppoſe a body to deſcribe the circumference lZp with the velocity acquired at V continued uniform. Then, when MN is diminiſhed without limit, the velocity with which it is deſcribed : the velocity with which xt is deſcribed :: $mx : VZ$ (Art. 291); therefore the time of deſcribing MN : the time of deſcribing xt :: $\frac{MN}{mx} : \frac{tx}{VZ} :: \frac{xr}{mx} : \frac{xt}{Vx}$ (Art. 15). Now, the triangles Vxm, xrt are ultimately ſimilar, and $Vx : mx :: xt : xr$; therefore $\frac{xr}{mx} = \frac{xt}{Vx}$; conſequently, the time of deſcent down MN, is equal to the time of deſcribing the correſponding circular arc xt with the velocity VZ; and the ſame may be proved of all other correſponding arcs in the cycloid and the circle; therefore the whole time of an oſcillation is equal to the time of deſcribing the ſemi-circumference lZp, with the velocity acquired at V continued uniform.

Prop. LXXIV.

293. *The time of an oſcillation in a cycloid is to the time of deſcent down it's axis, as the circumference of a circle to it's diameter.*

If a body fall down the chord OV, the velocity acquired at V is equal to the velocity in the cycloid at V (Art. 270); and with this velocity continued uniform, the body would deſcribe $2OV$, or VL, or Vl, in the time of deſcent down OV (Art. 237); that is, in the time of deſcent down DV (Art. 264). It



appears then, that the time of an oſcillation is equal to the time of deſcribing lZp with the velocity acquired in the cycloid at V (Art. 292); and that the time of deſcent down the axis DV is equal to the time of deſcribing Vl with the ſame velocity; therefore, the time of an oſcillation : the time of deſcent down the axis :: the time of deſcribing the circumference lZp, with the velocity VZ : the time of deſcribing Vl with the ſame velocity :: lZp : Vl (Art. 13) :: $2lZp$: $2Vl$ :: the circumference of a circle : it's diameter.

294. Cor. 1. The time of an oſcillation in a given cycloid, at a given place, is the ſame, whether the body oſcillate in a greater or a ſmaller arc.

For, the time of an oſcillation bears an invariable ratio to the time of deſcent down the axis, which, in a given cycloid, is given.

295. Cor. 2. The time of an oſcillation in a ſmall circular arc whoſe radius is CV, is to the time of deſcent down $\frac{1}{2}CV$, as the circumference of a circle to it's diameter.

For, the time of an oſcillation in this circular arc is equal to the time of an oſcillation in an equal arc of the cycloid AVB (Art. 288).

296. Cor. 3. The time of an oſcillation in a cycloid, or ſmall circular arc, when the force of gravity is given, varies as the ſquare root of the length of the ſtring.

For, the time of an oſcillation varies as the time of deſcent down half the length of the ſtring; that is, as the ſquare root of half the length of the ſtring, or as the ſquare root of it's whole length.

Ex. 1. To compare the times in which two pendulums vibrate, whoſe lengths are 4 and 9 inches.

Since $T \propto \sqrt{L}$, we have $T : t :: \sqrt{4} : \sqrt{9} :: 2 : 3$.

Ex. 2. If a pendulum, whoſe length is 39.2 inches, vibrate in one ſecond, in what time will a pendulum vibrate whoſe length is L inches?

$\sqrt{39.2} : \sqrt{L} :: 1 : T = \sqrt{\frac{L}{39.2}}$, the time required, in ſeconds.

Ex. 3. To compare the lengths of two pendulums, whoſe times of oſcillation are as 1 to 3.

Since $T \propto \sqrt{L}$, $T^2 \propto L$; therefore, $1 : 9 :: L : l$.

297. Cor. 4. The number of oſcillations, which a pendulum makes in a given time, at a given place, varies inverſely as the ſquare root of it's length.

Let n be the number of oſcillations, t the time of one oſcillation; then, nt is the whole time, which, by the ſuppoſition, is given; therefore, $n \propto \frac{1}{t}$ (Alg. Art. 206), and $t \propto \sqrt{L}$; conſequently, $n \propto \frac{1}{\sqrt{L}}$.

Ex. 1. If a pendulum, whoſe length is 39.2 inches, vibrate ſeconds, or 60 times in a minute, how often will a pendulum whoſe length is 10 inches vibrate in the ſame time?

Since $n \propto \frac{1}{\sqrt{L}}$, we have $\sqrt{10} : \sqrt{39.2} :: 60 : 60 \times \sqrt{3.92} = 118.8$, nearly, the number of oſcillations required.

Ex. 2. If a pendulum, whoſe length is 39.2 inches, vibrate ſeconds, to find the length of a pendulum which will vibrate double ſeconds, or 30 times in a minute.

Since $n \propto \frac{1}{\sqrt{L}}$, we have, $L \propto \frac{1}{n^2}$; and in this caſe, $\overline{30}|^2 : \overline{60}|^2 :: 39.2 : L = 4 \times 39.2 = 156.8$ inches, the length required.

Ex. 3. To find how much the pendulum of a clock which loſes one ſecond in a minute, ought to be ſhortened.

Since the pendulum vibrates 59 times, whilſt a pendulum of 39.2 inches vibrates 60 times, it's length may be found as in the laſt example; $\overline{59}|^2 : \overline{60}|^2 ::$ 39.2 : 40.5, it's length; and it ought to be 39.2 inches; therefore, 40.5 − 39.2, or 1.3 inches, is the quantity by which it ought to be ſhortened, in order that it may vibrate ſeconds.

298. Cor. 5. If the force of gravity be not given, the time of an oſcillation varies as the ſquare root of the length of the pendulum directly, and as the ſquare root of the force of gravity inverſely.

For, the time of an oſcillation varies as the time of deſcent down half the length of the ſtring; and in general, the time of deſcent through any ſpace $\propto \sqrt{\frac{S}{F}}$ (Art. 240); in this caſe, $S = \frac{1}{2}L$; therefore $S \propto L$, and the time of deſcent $\propto \sqrt{\frac{L}{F}}$; hence T, the time of an oſcillation, $\propto \sqrt{\frac{L}{F}}$.

299. Cor. 6. If the length of the pendulum be given, $T \propto \frac{1}{\sqrt{F}}$; and $F \propto \frac{1}{T^2}$.

The time in which a given pendulum vibrates, increaſes as it is carried from a greater latitude on the earth's ſurface to a leſs; therefore, the force of gravity decreaſes as the latitude decreaſes.

300. Cor. 7. The force of gravity at the equator : the force of gravity at any propoſed latitude :: the

length of a pendulum which vibrates ſeconds at the equator : the length of a pendulum which vibrates ſeconds at the propoſed latitude.

For, $T \propto \sqrt{\frac{L}{F}}$; if therefore T be given, $\sqrt{F} \propto \sqrt{L}$, or $F \propto L$.

301 Cor. 8. If the chord BV* be drawn, the time of deſcent down the cycloidal arc BV : the time of deſcent down the chord :: $DB : BV$.

For, the time of deſcent down the arc BV is equal to half the time of an oſcillation (Art. 272); therefore, the time of deſcent down the arc BV : the time of deſcent down DV :: half the circumference of a circle : it's diameter :: $DB : DV$; alſo, the time of deſcent down DV : the time of deſcent down the chord BV :: $DV : BV$; therefore, ex æquo, the time of deſcent down the arc BV : the time of deſcent down the chord :: $DB : BV$.

Prop. LXXV.

302. *The ſpace through which a body falls by the force of gravity in the time of an oſcillation in a cycloid, or ſmall circular arc, is to half the length of the pendulum, as the ſquare of the circumference of a circle to the ſquare of it's diameter.*

The ſpaces through which bodies fall by the action of the ſame uniform force are as the ſquares of the times (Art. 241); and ſince the time of an oſcillation : the time of deſcent down half the length of the pendulum :: the circumference of a circle : it's diameter, the ſpace fallen through in the time of an oſcillation :

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* The line BV is wanting in the figure.

half the length of the pendulum :: the ſquare of the circumference of a circle : the ſquare of it's diameter.

Ex. To find how far a body will fall by the force of gravity in one ſecond, where the length of the pendulum, which vibrates ſeconds, is 39.2 inches.

The circumference of a circle : it's diameter :: 3.14159 : 1; conſequently, the ſpace fallen through in one ſecond : $\frac{39.2}{2}$ :: $\overline{3.14159}|^2 : 1^2$; hence, the ſpace fallen through is $19.6 \times \overline{3.14159}|^2 = 193$ inches, or $16\frac{1}{12}$ feet, nearly.

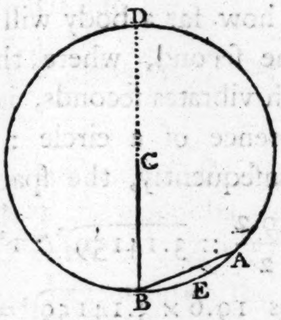
If the arc, in which a body oſcillates, be diminiſhed, the effect of the air's reſiſtance is diminiſhed; and when the arc is very ſmall, this reſiſtance does not ſenſibly diminiſh the time of an oſcillation. By obſerving, therefore, the length of a pendulum which vibrates ſeconds in very ſmall arcs, we determine the ſpace through which a body would fall in vacuo in one ſecond, with ſufficient accuracy for all practical purpoſes.

Prop. LXXVI.

303. *The time of deſcent to the loweſt point in a ſmall circular arc is to the time of deſcent down it's chord, as the circumference of a circle to four times the diameter.*

Let AB be the arc, C it's center, BD the diameter perpendicular to the horizon; T the time of deſcent down the arc AEB, t the time of deſcent down the chord, C the circumference of a circle, D it's diameter. Then, $2T$ is the time of an oſcillation of the pendulum CB; therefore, $2T$: the time down $\frac{CB}{2}$ :: $C : D$ (Art. 295);

295); and the time down $\frac{CB}{2}$: the time down DB,



or AB, :: 1 : 2 (Art. 241); therefore, $2T$: the time down AB (t) :: $C : 2D$, and $T : t :: C : 4D$.

PROP. LXXVII.

304. *The force which accelerates or retards a body's motion in a cycloid varies as the arc intercepted between the body and the loweſt point.*

Let DV repreſent the whole force of gravity, (See Fig. in p. 166.) from P draw PH parallel to AD meeting the circle DHV in H; join DH, HV.

Then, the whole force DV, which acts upon the body at P, may be reſolved into the two DH, HV; of which, DH is in the direction of the ſtring, and therefore neither accelerates nor retards the motion of P; and HV is in the direction of the tangent at P (Art. 281), and therefore wholly employed in accelerating or retarding the motion in the curve; conſequently, the force of gravity : the accelerating force :: DV : HV; and ſince the force of gravity, and DV are invariable, the accelerating force $\propto HV \propto 2HV \propto PV$ (Art. 283).

305. Cor. 1. If a body move in any line, and be acted upon by a force which varies as the diſtance from the loweſt point, the motion of this body will be ſimilar to the motion of a body oſcillating in a cycloid.

For, if an arc, meaſured from the vertex of a cycloid, be taken equal to the line, and the accelerating forces, in the line and the cycloid, at theſe equal diſtances from the loweſt points, be equal, they will always be equal, becauſe they vary according to the ſame law; and the bodies, being impelled by equal forces, will be equally accelerated, and deſcribe equal ſpaces in any given time.

306. Cor. 2. The time of deſcent to the loweſt point in the line will always be the ſame, from whatever place the body begins to fall.

307. Cor. 3. If the diſtance of the body from the loweſt point at the beginning of the deſcent, be made the radius, the velocity acquired will be repreſented by the right ſine, and the time by the arc, whoſe verſed ſine is the ſpace fallen through.

Prop. LXXVIII.

308. *If a body vibrate in a circular arc, the force which accelerates or retards it's motion varies as the ſine of it's diſtance from the loweſt point.*

Let a body oſcillate in a circular arc whoſe radius is AC; from the center C, and A the place of the body, draw CB, AE perpendicular to the horizon, and take AE to repreſent the force of gravity; draw AD perpendicular to CB, and EF perpendicular

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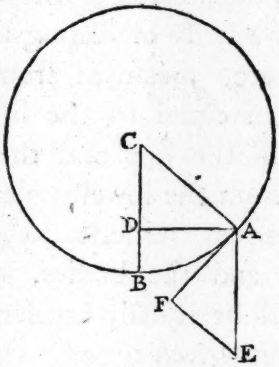
cular to *AF*, which is a tangent to the circle at *A*.



Then, the force *AE* is equivalent to the two *AF*, *FE*; of which *FE* is perpendicular to the tangent *AF*, or in the direction of the radius *CA*, and can neither accelerate nor retard the motion of the body; the other, *AF*, is in the direction of the tangent, and is wholly employed in accelerating or retarding the body's motion; therefore, the force of gravity : the accelerating force :: *AE* : *AF*, that is, from the ſimilar triangles *AEF*, *CAD*, :: *CA* : *AD*; and conſequently, the accelerating force $= \frac{\text{gravity} \times AD}{AC}$; in which expreſſion, gravity and the radius *AC* are invariable; therefore, the accelerating force varies as *AD*.

309. Cor. 1. If the accelerating force were proportional to the arc, the oſcillations, whether in greater or ſmaller arcs, would be performed in equal times (Art. 306); but, ſince the ſine does not increaſe as faſt as the arc, the force in the greater arc is leſs than that which would be ſufficient to make the time of oſcillation equal to the time in the ſmaller arc;

therefore, the time of oſcillation in the greater circular arc is greater than in the leſs.

310. Cor. 2. Call F the force in the direction AE, then ſince AC is invariable, the accelerating force in the curve $\propto F \times AD$; and if $F \times AD \propto AB$, or $F \propto \frac{AB}{AD}$, the accelerating force varies as AB, and the times of oſcillation in different arcs are equal (Art. 306).

SCHOLIUM.

311. In this Section we have conſidered the vibrations of a ſimple pendulum only, or of a ſingle particle of matter, ſuſpended by a ſtring, the gravity of which is neglected. The propoſitions are indeed applicable in practice, when the diameter of the body is ſmall with reſpect to the length of the ſtring by which it is ſuſpended, and the weight of the ſtring inconſiderable when compared with the weight of the body. That the concluſions are not ſtrictly true in this caſe, is evident from the conſideration that two particles of matter, at different diſtances from the axis of ſuſpenſion, do not vibrate in the ſame time (Art. 296); and conſequently, that when they are connected together, they affect each other's motion; thus, the time of vibration of the two particles when united, is different from the time in which either would vibrate alone.

The method of determining the time of vibration of a compound pendulum, the Reader will find in the *Principles of Fluxions*, Art. 63; to which place he is alſo referred for the inveſtigation of the rules for determining the centers of *Gyration*, and *Percuſſion*; queſtions properly belonging to Mechanics, but inſerted in that part of the Work, becauſe the rules cannot eaſily be applied to the determination of thoſe points, even in the moſt ſimple caſes, without the aſſiſtance of the fluxional calculus.

312. To avoid the introduction of analytical demonſtrations in ſubjects profeſſedly geometrical,

Sir I. NEWTON and other Writers, have had recourſe to indefinitely ſmall or evaneſcent increments, which continually approximate to the true increments of the quantities whoſe finite values are required. This method may be applied with ſucceſs in all caſes where the *difference* between the *aſſumed* and the *true* increments continually decreaſes, and at length vaniſhes, with reſpect to the increments themſelves; or, which amounts to the ſame thing, when the ratio which the ſum of the differences bears to one of the increments, does not exceed a finite ratio: for, by obſerving the *limit* to which the ſum of the aſſumed increments approaches, when their number is increaſed and their magnitudes are diminiſhed in infinitum, it is evident that the ſum of the real increments is obtained. In the ſame manner, when there are two ranks of quantities, in which the aſſumed increments continually approximate to the real increments, as in the former inſtance, and the limiting ratio of the ſums of the aſſumed increments in theſe caſes, when their numbers are increaſed and their magnitudes diminiſhed without limit, is obtained, the exact ratio of the quantities themſelves is obtained. Theſe propoſitions are laid down by Sir I. NEWTON in the firſt Section of the Principia, Lem. 3d. and 4th. and the ſame mode of reaſoning has been applied in Art. 292 to compare the time of an oſcillation in the cycloid BVA, with the time of deſcribing the arc lZp with the velocity acquired at V continued uniform. In this Art. it is ſuppoſed that the time of deſcribing MN, with the *uniform* velocity mx, is the increment of the former time, and that the time of deſcribing xt, the ſide of a triangle *ſimilar* to Vxm, with the velocity VZ, is the increment of the latter; theſe *aſſumed* increments, it is manifeſt, differ from the true increments of the times under conſideration; but when they are diminiſhed without limit, they differ from them by quantities which are evaneſcent with reſpect to the whole increments, and therefore by determining the limiting ratio of the ſums of the aſſumed increments, we obtain the ratio of the actual times of deſcribing the correſponding arcs.

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SECTION IX.

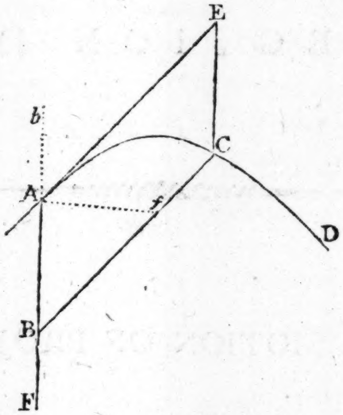
ON THE MOTION OF PROJECTILES.

Prop. LXXIX.

313. *A BODY projected in any direction, not perpendicular to the horizon, will deſcribe a parabola, on ſuppoſition that the force of gravity is uniform, and acts in parallel lines, and that the motion is not affected by the reſiſtance of the air.*

Let a body be projected from *A* in the direction *AE*, from which point draw *ABF* perpendicular to the horizon; alſo, let *AE* be the ſpace over which the velocity of projection would carry the body in any time *T*, and *AB* the ſpace through which the force of gravity would cauſe it to deſcend in the ſame time; complete the parallelogram *AC*; then, in conſequence of the two motions, the body will be found in *C* at the end of that time. For, the motion in the direction

AE can neither accelerate nor retard the approach of the body to the line *BC* (Art. 29); therefore, at the end of the time *T*, the body will be in the line *BC*;



and, by the ſame mode of reaſoning, it appears that it will be in the line *EC* at the ſame time; conſequently, it will be at *C*, the point of their interſection, at the end of the time *T*. Now, ſince *AE* is the ſpace which would be deſcribed in the time *T*, with the velocity of projection continued uniform, $AE \propto T$ (Art. 13); and $BC = AE$; therefore $BC \propto T$, and $BC^2 \propto T^2$. Alſo, ſince *AB* is the ſpace through which the body would fall by the force of gravity in the time *T*, $AB \propto T^2$ (Art. 241); hence, $AB \propto BC^2$; and this is the property of a parabola, in which *AF* is a diameter, and *BC* an ordinate to the abſciſſa *AB*.

314. Cor. 1. The axis, and all the diameters of the parabola deſcribed, being parallel to *AF*, are perpendicular to the horizon.

315. Cor. 2. The direction of projection *AE* is parallel to the ordinate *BC*, and therefore it is a tangent to the curve at *A*.

316. Cor. 3. The time in which a body deſcribes the arc *AC* is equal to the time in which it would fall from *A* to *B* by the force of gravity, or deſcribe *AE* with the firſt velocity continued uniform.

317. Cor. 4. If the $\angle EAf$ be made equal to the $\angle EAb$, the line *Af* will paſs through the focus of the parabola deſcribed.

318. Cor. 5. The body will always be found in the plane *ACB* which paſſes through the direction of projection and the perpendicular to the horizon.

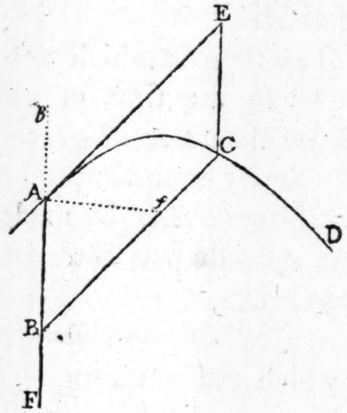
319. Cor. 6. If the motion in the direction *AE* be produced by the action of an uniform force, $AE \propto T^2 \propto AB$; or $AB \propto BC$; therefore, the locus of the point *C* is a right line.

PROP. LXXX.

320. *The velocity of the projectile, at any point in the parabola, is ſuch as would be acquired in falling through one fourth part of the parameter belonging to that point.*

Let *AB* be the ſpace through which a body muſt fall by the force of gravity to acquire the velocity of the projectile at *A*; and *AE* the ſpace deſcribed with that velocity continued uniform, in the time of falling through *AB*; then $2AB = AE$ (Art. 237); and, completing the parallelogram *AC*, $2AB = BC$; hence, $4AB^2 = BC^2$. Alſo, ſince *C* is a point in the parabola,

bola, and BC an ordinate to the abſciſſa AB (Art. 313),



if P be the parameter belonging to the point A, $P \times AB = BC^2 = 4AB^2$; therefore $AB = \frac{1}{4}P$.

321. Cor. 1. If the velocity at A be given, the parameter at that point is the ſame, whatever be the direction of the body's motion.

322. Cor. 2. If AB be the ſpace through which a body muſt fall to acquire the velocity at A, and a circle be deſcribed from the center A with the radius AB, the focus of the parabola deſcribed will lie in the circumference of this circle, whatever be the direction of projection; ſince the diſtance of any point in the parabola from the focus, is one fourth part of the parameter belonging to that point.

323. Cor. 3. The velocity at any point in the parabola varies as the ſquare root of the parameter belonging to that point.

Since the velocity is ſuch as would be acquired in falling through $\frac{1}{4}P$, it varies as $\sqrt{\frac{1}{4}P}$ (Art. 241), or as $\sqrt{P}$.

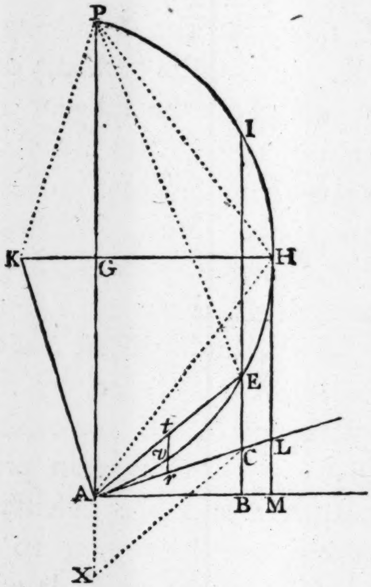
324. Cor. 4. The velocity is the leaſt in the vertex of a parabola; and at equal diſtances from the vertex the velocities are equal.

For, the parameter belonging to the vertex is the leaſt; and the parameters at equal diſtances from the vertex are equal.

Prop. LXXXI.

325. *To find the direction in which a body muſt be projected from a given point, with a given velocity, to hit a given mark.*

Let *A* be the point from which the body is to be projected, *C* the given mark; join *AC*, and from *A* draw



draw *AB* parallel, and *AP* perpendicular to the horizon; take *AP* equal to four times the ſpace through which a body muſt fall to acquire the given velocity of projection, (determined by Art. 248, Caſe 3); then will *AP* be the parameter belonging to the point *A* of the parabola deſcribed (Art. 320.) Draw *AK* perpendicular to *AC*; biſect *PA* in *G*, and draw *KGH* perpendicular to *AP*, meeting *AK* in *K*; join *KP*; then the triangles *KGP*, *KGA* being ſimilar and equal, *KP*=*KA*. From *K* as a center with the radius *KA*, or *KP*, deſcribe a circle *AHP*, cutting *KGH* in *H*; through *C* draw *CEI* parallel to *AP*, and cutting the circle in *E* and *I*; join *AE*, *AI*; and if a body be projected, with the given velocity, in the direction *AE* or *AI*, it will hit the mark *C*.

Let the body be projected in the direction *AE*; join *PE*, and complete the parallelogram *AECX*; then *AX* is a diameter of the parabola deſcribed; and *XC*, which is parallel to the tangent *AE*, is in the direction of an ordinate to the abſciſſa *AX*; if then *XC* be the length of the ordinate to this abſciſſa, *C* is a point in the parabola. Now, ſince the angles *AEC*, *EAP* are alternate angles, and the angle *EAC*=the angle *EPA*, becauſe *AC* is a tangent to the circle at *A* (Euc. 16. 3), the triangles *EPA*, *EAC* are ſimilar; and *AP* : *AE* :: *AE* : *EC*; or, by ſubſtituting for *AE* and *EC* their equals *XC* and *AX*, *AP* : *XC* :: *XC* : *AX*; that is, *XC* is a mean proportional between the parameter and the abſciſſa; and therefore it is the ordinate belonging to that abſciſſa; hence, *C* is a point in the parabola which the body deſcribes.

In the ſame manner it may be ſhewn that, if the body be projected with the ſame velocity in the direction AI, it will hit the mark C.

326. Cor. 1. Join AH, HP; then the angle HAP $=$ the angle $HPA =$ the angle HAC.

327. Cor. 2. Becauſe KH is drawn through the center of the circle perpendicular to the chord EI, it biſects it, and conſequently it biſects the arc IHE (Euc. 30, 3); therefore, the angle $IAH =$ the angle HAE. That is, the two directions AE, AI make equal angles with AH, which biſects the angle PAC.

328. Cor. 3. Draw HLM touching the circle in H; then, when the point C coincides with L, the two directions AE, AI, coincide with AH.

329. Cor. 4. If the point C be taken in the plane AL, beyond L, the line CEI will not meet the circle. In this caſe, the velocity of projection is not ſufficient to carry the body to the diſtance AC.

330. Cor. 5. Biſect AC in r, and draw tvr parallel to AP; then tvr is in the direction of a diameter, to which AC is a double ordinate. Alſo, rt is the ſubtangent; and if rt be biſected in v, this point is in the parabola.

331. Cor. 6. A tangent to the parabola at v is parallel to the ordinate AC; therefore, v is the point in the parabola which is at the greateſt diſtance from AC*.

332. Cor. 7. The greateſt height of the projectile above the plane, meaſured in the direction of gravity, is $\frac{1}{4}EC$. For, $rv = \frac{1}{2}rt$, and $rt = \frac{1}{2}EC$; therefore, $rv = \frac{1}{4}EC$.

Prop.

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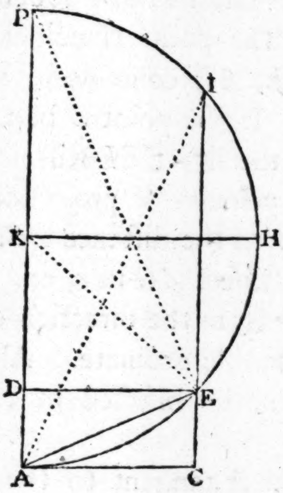
* The properties of the parabola here referred to, may be found in any Treatiſe of Conic Sections.

Prop. LXXXII.

333. *Having given the velocity and direction of projection, to find where the body will ſtrike the horizontal plane which paſſes through the point of projection.*

Let AC (Art. 325) coincide with the horizontal line AB; then AK coincides with AG; and PHA is a ſemicircle; alſo, HAC is an angle of 45°, and AE, AI, are equally inclined to AH.



From the laſt propoſition it appears, that if the velocity of projection be ſuch as would be acquired in falling through $\frac{1}{4}PA$, and AE, or AI, be the direction of projection, the range is AC. From E draw ED parallel to AC, or perpendicular to AP; join EK; then ED, or it's equal AC, is the ſine of the $\angle EKA$ to the radius KA; and the angle $EKA = 2\angle EPA = 2\angle EAC$;

therefore, AC is the ſine of $2\angle EAC$ to the radius KA; and the ſine of a given angle is proportional to the radius; conſequently, rad. : KA :: ſin. $2\angle EAC$: AC; hence, $AC = \frac{\text{ſin. } 2\angle EAC \times KA}{\text{rad.}}$. If V be taken to repreſent the velocity of projection, P the parameter AP, and $m = 16\frac{1}{12}$, then $AC = \frac{\text{ſin. } 2\angle EAC \times P}{2\text{ rad.}} = \frac{\text{ſin. } 2\angle EAC \times V^2}{2\text{ rad.} \times m}$ (Art. 248).

In the ſame manner, if AI be the direction of projection, $AC = \frac{\text{ſin. } 2\angle IAC \times P}{2\text{ rad.}} = \frac{\text{ſin. } 2\angle IAC \times V^2}{2\text{ rad} \times m}$.

334. Cor. 1. Hence, $AC \propto \text{ſin. } 2\angle EAC \times V^2$.

335. Cor. 2. If the velocity of projection be invariable, the horizontal range varies as ſin. $2\angle EAC$.

336. Cor. 3. The range is the greateſt when ſin. $2\angle EAC$ is the greateſt; that is, when the $\angle EAC$ is 45°. In this caſe, $AC = \frac{\text{rad.} \times P}{2\text{ rad.}} = \frac{1}{2}P$.

337. Cor. 4. If the angle EAC be 15°, or 75°, ſin. $2\angle EAC = \frac{1}{2}$ rad. and $AC = \frac{1}{4}P$.

338. Cor. 5. If the range and the parameter be given, the angle of elevation may be found.

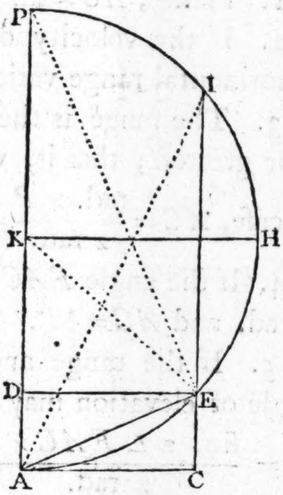
For, $AC = \frac{\text{ſin. } 2\angle EAC \times P}{2\text{ rad.}}$; therefore, ſin. $2\angle EAC = \frac{2AC \times \text{rad.}}{P}$.

339. Cor. 6. If AC and the angle EAC be known, $P = \frac{\text{ſin. } 2\angle EAC}{2AC \times \text{rad.}}$; and $V^2 = \frac{2\text{ rad.} \times mAC}{\text{ſin. } 2\angle EAC}$.

Prop. LXXXIII.

340. *The ſame things being given, to find the time of flight.*

The time in which the velocity of projection, V, would be acquired, or the time of deſcent down $\frac{1}{4}PA$, is $\frac{V}{2m}''$ (Art. 248); hence, the double of this time, or the time of deſcent down PA, is $\frac{V}{m}''$ (Art. 241). Let T be the time of deſcent down PA, t the time of deſcent down EC, or the time of flight (Art. 316); then $T^2 : t^2 :: PA : EC$; and from the ſimilar triangles PAE, AEC, $PA : AE :: AE : EC$; hence, $PA^2 : AE^2 ::$



$PA : EC :: T^2 : t^2$, and $PA : AE :: T : t$. Alſo, PA AE :: rad. : ſin. $\angle EPA$ :: rad. : ſin. $\angle EAC$; therefore $T : t$ :: rad. : ſin. $\angle EAC$, and $t = \frac{\text{ſin.} \angle EAC \times T}{\text{rad.}}$

$$= \frac{\text{ſin.} \angle EAC \times V}{\text{rad.} \times m}.$$

341. Cor. 1. If the velocity of projection be invariable, the time of flight $\propto$ ſin. $\angle EAC$.

342. Cor. 2. Hence, the time of flight is the greateſt, when ſin. $\angle EAC$ is the greateſt. In this caſe, the time becomes $\frac{\text{rad.}\times T}{\text{rad.}}$, or T; that is, the greateſt time of flight is equal to the time of deſcent down the parameter.

PROP. LXXXIV.

343. *The ſame things being given, to find the greateſt height to which the projectile riſes above the horizontal plane.*

The greateſt height is $\frac{1}{4}EC$, or $\frac{1}{4}AD$; and AD is the verſed ſine of the $\angle AKE$, or $2\angle EAC$, to the radius AK; conſequently, ſince the verſed ſine of a given angle varies as the radius, rad. : AK $(\frac{1}{2}P)$ :: the verſed ſine of $2\angle EAC$: AD. Hence, $AD = \frac{\text{ver. ſin. } 2\angle EAC \times P}{2 \text{ rad.}}$; and $\frac{1}{4}AD$, the greateſt height,

$$= \frac{\text{ver. ſin. } 2\angle EAC \times P}{8 \text{ rad.}} = \frac{\text{ver. ſin. } 2\angle EAC \times V^2}{8 \text{ rad.} \times m}.$$

344. Cor. 1. The greateſt height $\propto$ ver. ſin. $2\angle EAC \times V^2$; and when V is given, the greateſt height $\propto$ ver. ſin. $2\angle EAC$.

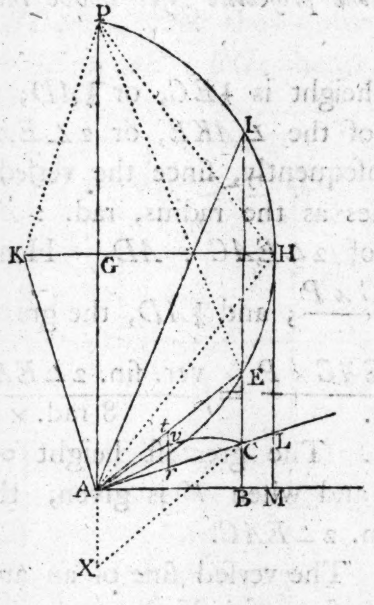
345. Cor. 2. The verſed ſine of an arc varies as the ſquare of the ſine of half the arc; therefore the greateſt height $\propto \overline{\text{ſin.}\,\angle EAC}|^2 \times V^2$.

346. Cor. 3. A body, projected with a given velocity, will riſe to the greateſt height above the horizontal plane, when the angle of elevation EAC is a right angle. In this caſe, ver. ſin. $2\angle EAC = 2$ rad. and the greateſt altitude is $\frac{2 \text{ rad.} \times P}{8 \text{ rad.}} = \frac{P}{4}$.

Prop. LXXXV.

347. *The velocity and direction of projection being given, to find where the body will ſtrike a given inclined plane which paſſes through the point of projection.*

It appears from Art. 325, that if a body be projected from *A*, in the direction *AE*, with the velocity acquired in falling down $\frac{1}{4}PA$, it will ſtrike the



plane *AC* in the point *C*. Let *I* be the angle of inclination *CAB*; *E* the angle of elevation *EAC*; *Z* the angle *EAP*. Then, in the triangle *EAP*, $AE : AP ::$ ſin. $\angle EPA$: ſin. $\angle AEP$; and the $\angle EPA =$ the $\angle EAC = E$; alſo the $\angle AEP =$ the $\angle ECA =$ the ſupplement of the $\angle ACB$; hence, $AE : AP ::$ ſin. E : coſ. I; therefore, $AE = \frac{\text{ſin. } E \times AP}{\text{coſ. } I}$.

Again, in the triangle EAC, $AC : AE :: \text{ſin. } \angle AEC$ ($\text{ſin. } Z$) : $\text{ſin. } \angle ACE$ ($\text{coſ. } I$); therefore, $AC = \frac{\text{ſin. } Z \times AE}{\text{coſ. } I}$; and by ſubſtituting for AE it's value $\frac{\text{ſin. } E \times AP}{\text{coſ. } I}$, $AC = \frac{\text{ſin. } E \times \text{ſin. } Z \times AP}{\overline{\text{coſ. } I}\rvert^2} = \frac{\text{ſin. } E \times \text{ſin. } Z \times V^2}{\overline{\text{coſ. } I}\rvert^2 \times m}$.

348. Cor. Hence, $AC \propto \frac{\text{ſin. } E \times \text{ſin. } Z \times V^2}{\overline{\text{coſ. } I}\rvert^2}$.

Prop. LXXXVI.

349. *The ſame things being given, to find the time of flight.*

Let T be the time of deſcent down PA, t the time of deſcent down EC or the time of flight; then, as in Art. 340, $T^2 : t^2 :: PA : EC$; and ſince, in the ſimilar triangles PAE, AEC, $PA : AE :: AE : EC$, we have $PA^2 : AE^2 :: PA : EC :: T^2 : t^2$; and $PA : AE :: T : t$; but $PA : AE :: \text{ſin. } \angle PEA : \text{ſin. } \angle EPA :: \text{ſin. } \angle ECA : \text{ſin. } \angle EAC :: \text{coſ. } I : \text{ſin. } E$; therefore, $T : t :: \text{coſ. } I : \text{ſin. } E$, and $t = \frac{\text{ſin. } E \times T}{\text{coſ. } I} = \frac{\text{ſin. } E \times V}{\text{coſ. } I \times m}$.

350. Cor. Hence, $t \propto \frac{\text{ſin. } E \times V}{\text{coſ. } I}$; and if V be invariable, $t \propto \frac{\text{ſin. } E}{\text{coſ. } I}$.

Prop. LXXXVII.

351. *The ſame things being given, to find the greateſt height of the projectile above the plane* AC, *meaſured in the direction of gravity.*

The greateſt height is $\frac{1}{4}EC$ (Art. 332;) and in the triangle AEC, $EC : AE :: \text{ſin. } E : \text{coſ. } I$; therefore, $EC = \frac{\text{ſin. } E \times AE}{\text{coſ. } I}$; and, by ſubſtituting for AE it's value $\frac{\text{ſin. } E \times AP}{\text{coſ. } I.}$ (Art. 347), we have $CE = \frac{\overline{\text{ſin. } E}|^{2} \times AP}{\overline{\text{coſ. } I}|^{2}}$; and $\frac{1}{4}EC = \frac{\overline{\text{ſin. } E}|^{2} \times AP}{4\,\overline{\text{coſ. } I}|^{2}} = \frac{\overline{\text{ſin. } E}|^{2} \times V^{2}}{\overline{\text{coſ. } I}|^{2} \times 4m}$, the greateſt height required.

352. Cor. The greateſt height varies as $\frac{\overline{\text{ſin. } E}|^{2} \times V^{2}}{\overline{\text{coſ. } I}|^{2}}$.

SCHOLIUM.

353. The theory of the motion of projectiles, given in this ſection, depends upon three ſuppoſitions, which are all inaccurate; 1ſt. that the force of gravity, in every point of the curve deſcribed, is the ſame; 2d. that it acts in parallel lines; 3d. that the motion is performed in a non-reſiſting medium. The two former of theſe, indeed, differ inſenſibly from the truth. The force of gravity, without the Earth's ſurface, varies inverſely as the ſquare of the diſtance from the center; and the altitude to which we can project a body from the ſurface is ſo ſmall, that the variation of the force, ariſing from the alteration of the diſtance from the center of the Earth, may ſafely be neglected. The direction of the force is every where perpendicular to the horizon; and if perpendiculars be thus drawn, from any two points in the curve which we can cauſe a body to deſcribe, they may be conſidered as parallel, ſince they only meet at, or nearly at, the center of the Earth. Even the reſiſtance of the air does not materially affect the motions of heavy bodies, when they are projected with ſmall velocities. In other caſes, however, this reſiſtance is ſo great as to render the concluſions drawn from the theory almoſt entirely inapplicable in practice. From experiments made to determine the motions of cannon

balls, it appears that when the initial velocity is conſiderable, the air's reſiſtance is 20 or 30 times as great as the weight of the ball; and that the horizontal range is often not $\frac{1}{20}$ part of that which the preceding theory gives. It appears alſo, that when the angle of elevation is given, the horizontal range varies nearly as the ſquare root of the velocity of projection; and the time of flight as the range; whereas, according to the theory, the time varies as the velocity, and the range as the ſquare of the velocity of projection (Arts. 340. 334). Theſe experiments, made with great care, and by men of eminent abilities, ſhew how little the parabolic theory is to be depended upon in determining the motions of military projectiles. See ROBINS's *New Theory of Gunnery*, and HUTTON's *Mathematical Dictionary*, article *Gunnery*.

Beſides diminiſhing the velocity of the projectile, the air's reſiſtance will alſo change it's direction, whenever the body has a rotatory motion about an axis which does not coincide with the direction in which it is moving. For the velocity of that ſide of the body, on which the rotatory and progreſſive motions conſpire, is greater than the velocity of the other ſide, where they are contrary to each other; and therefore the reſiſtance of the air, which increaſes with the velocity, will be greater in the former caſe than in the latter, and cauſe the body to deviate from the line of it's motion; this deviation will alſo be from the *plane* of the firſt motion, unleſs the axis of rotation be perpendicular to that plane.

Upon this principle Sir I. NEWTON explains the irregular motion of a tennis ball*, and the ſame cauſe has been aſſigned by Mr. **Robins** for the deviation of a bullet from the vertical plane*. Mr. **Euler**, indeed, in his remarks on the *New Theory of Gunnery*, contends that the reſiſtance of the air can neither be increaſed nor diminiſhed by the rotation of the ball; becauſe ſuch a motion can produce no effect but in the direction of a tangent to the ſurface of the revolving body; and the tangential force, he affirms, is almoſt entirely loſt. In this inſtance, the learned Writer ſeems to have been miſled by the common theory of reſiſtances, according to which the tangential force produces no effect; whereas, from experiments lately made, with a view to aſcertain the quantity and laws of the air's reſiſtance, it appears that every theory which neglects the tangential force muſt be erroneous.

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* Phil. Tranſ. Vol. VI. p. 3078. MACLAURIN's NEWTON, p. 120.

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* Tracts, Vol. I. p. 151, 198, 214.

THE END OF PART I. VOL. III.

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